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A Semiring-Based Hierarchical Decision Model and Application

Fujun Hou

School of Management and Economics, Beijing Institute of Technology, Beijing 100081, P.R. China

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Abstract: Pairwise comparison matrix (PCM) is widely used in multi-criteria decision making (MCDM). A criterion for acceptable consistency of PCM is ¯rstly intro-duced in the present paper. Then, based on the author's previous work on the iso-morphism investigation of PCMs, a Multiplicative-Pairwise-Comparison-Based Hi-erarchical Decision Model (MPCbHDM) is proposed with the following approaches included: (1) the newly introduced criterion is used to check acceptable consistency; (2) the row's geometric mean method is used for deriving the local weights; (3) a proposed Hierarchy Composition Rule is used for computing the sub-criterion's global weights; and (4) the weighted geometric mean method is used as the aggre-gation rule, where the alternative's local weights are min-normalized. The MPCb-HDM has the property of preserving rank. Moreover, it has counterparts in the fuzzy case. Finally, the MPCbHDM is applied to a conference site selection problem with a computer-based software.

Keywords: Pairwise comparison, fuzzy preference relations, hierarchical decision model, isomorphism

1. Introduction

Pairwise comparison has proven to be a powerful tool in multi-criteria decision making (MCDM). The analytic hierarchy process (AHP) is a hierarchical structure decision model proposed by Saaty (1980), where the multiplicative pairwise comparison matrix (PCM) is used as the preference information.

Ever since the AHP was introduced, this technique has been widely used by practitioners and extensively investigated by researchers (see, Bana et al. (2008); Dodd and Donegan (1995); Farkas; Ishizaka and Labib (2011); Triantaphyllou (2001), and the references therein). However, AHP has also suffered from a number of controversies among which the rank reversal has attracted everlasting attentions of the researchers (Belton and Gear, 1983; Dyer, 1990; Maleki and Zahir, 2012).

There are usually two kinds of PCMs used in the decision practice, one is the multiplicative PCM (Saaty, 1980), and, the other the fuzzy preference relations (FPRs, for short) (Chang and Wang, 2009; Chiclana et al., 2001; Tanino, 1984; Wang and Chen, 2007). There exists a semiring-sense isomorphic relation between a consistent multiplicative PCM and a consistent FPRs (Hou, 2011, 2012).

In this paper, we first introduce a criterion for acceptable consistency of PCM, which is independent of the scale and can be intuitively interpreted. Then, based on our previous work on the isomorphism investigation of the PCMs (Hou, 2011, 2012), a multiplicative Pairwise Comparison based Hierarchical Decision Model (MPCbHDM) is proposed. The MPCbHDM has the property of preserving rank. Moreover, it has counterparts in the fuzzy case.

Section 2 covers a concise overview of two kinds of pairwise comparison matrix and some counterpart results in a semiring framework including the method for deriving weights, min-normalization and hierarchical aggregation rule. Section 3 contains the proposed acceptable criterion for PCMs. The main steps of the proposed hierarchical model is presented in Section 4. An illustrative example is included in Section 5 and Section 6 contains our concluding remarks. Some semiring related results are included as an appendix.

2. Preliminary

We recall some results of PCMs obtained in semiring frameworks, see Hou (2011, 2012) for details.

2.1 *Multiplicative PCM and fuzzy preference relations (FPRs)*

E-mail: houfj@bit.edu.cn

Denote by $X=\{x_1,x_2,\ldots,x_n\}$ the object set and $I=\{1,2,\ldots,n\}$ the index set. Let i,j,k and l be index variables. Let α be a real number and $\alpha\in(1,+\infty)$.

For a n by n PCM $\mathbf{M}_{n\times n}=\left(p_{jk}\right)_{n\times n}$ with $\forall j,k\left(p_{jk}\in[1/\alpha,\alpha]\right)$, the conditions of reciprocity and consistency (multiplicative sense) are given by $\forall j,k\left(p_{jk}p_{kj}=1\right)$ and $\forall j,k,l\left(p_{jl}p_{lk}=p_{jk}\right)$, respectively. For a consistent multiplicative PCM $\mathbf{M}_{n\times n}=\left(p_{jk}\right)_{n\times n}$, a $n\times 1$ vector $W=\left(\omega_j\right)_{n\times 1}$ with $\forall j\left(\omega_j\in[1/\alpha,\alpha]\right)$, is called a weight vector of $\mathbf{M}_{n\times n}$ such that $\forall j,k\in I, p_{jk}\omega_k=\omega_j$.

Likewise, for a n by n FPRs $\mathbf{A}_{n\times n}=\left(a_{jk}\right)_{n\times n}$ with $\forall j,k\left(a_{jk}\in[0,1]\right)$, the conditions of reciprocity and consistency (fuzzy sense) are given by $\forall j,k\left(a_{jk}+a_{kj}=1\right)$ and $\forall j,k,l\left(a_{jl}+a_{lk}-0.5=a_{jk}\right)$, respectively. For a consistent FPRs $\mathbf{A}_{n\times n}=\left(a_{jk}\right)_{n\times n}$, a $n\times 1$ vector $V=\left(v_j\right)_{n\times 1}$ with $\forall j\left(v_j\in[0,1]\right)$, is called a weight vector of $\mathbf{A}_{n\times n}$ such that $\forall j,k\in I, a_{jk}+v_k-0.5=v_j$.

The above knowledge can be found in (Saaty, 1980; Tanino, 1984). They can be described in semiring senses as provided in Appendix A.2.

2.2 *Method for deriving priorities*

It is proved that, under a semiring framework, the mean method can be used for deriving priorities from a consistent PCM (Hou, 2012).

In particular, the priority vector corresponding to a consistent multiplicative PCM $\mathbf{M}=\left(p_{jk}\right)$ can be elicited by a geometric row mean method as

$$w_j=\left(\prod_{k=1}^{n}p_{jk}\right)^{\frac{1}{n}},\ j=1,2,\ldots,n. \qquad (1)$$

Correspondingly, the priority vector corresponding to a consistent PFRs $\mathbf{A}=\left(a_{jk}\right)$ can be elicited by an arithmetic row mean method as

$$v_j=\frac{1}{n}\left(\sum_{k=1}^{n}a_{jk}\right),\ j=1,2,\ldots,\mathrm{n}. \qquad (2)$$

For deriving priorities from consistent PCMs, different methods provide indifferent results. For inconsistent PCMs, any a method provides an approximate vector in a sense near to the 'real' priorities of the PCM. Moreover, the Eq.(2) is the isomorphism of Eq.(1) under the semiring mapping $\phi(x)=0.5\left(1+\log_\alpha x\right)$. Hence, the row mean method can be used to obtain priorities from a PCM.

2.3 *Min-normalization*

Vector normalization is a widely used technique so as to obtain uniqueness. The min-normalization under a semiring framework is proposed in (Hou, 2011).

Proposition 2.1 If a multiplicative PCM is consistent, then there exists one and only one multiplicatively min-normalized vector (a multiplicatively min-normalized vector is a vector with all entries in $[1,\alpha]$ and at least one entry being 1) as its priority vector which is denoted by

$$\bar{\omega}_i^{\min}=\omega_i\Big/\min_k\left\{\omega_k\right\} \qquad (3)$$

Proposition 2.2 If a PFRs is consistent, then there exists one and only one additively min-normalized vector (an additively min-normalized vector is a vector with all entries in $[0.5,1]$ and at least one entry being 0.5) as its priority vector which is denoted by

$$\bar{v}_i^{\min}=v_i-\min_k\left\{v_k\right\}+0.5 \qquad (4)$$

The Eq.(4) is the semiring-sense counterpart of Eq.(3) under the semiring mapping $\phi(x)=0.5\left(1+\log_\alpha x\right)$.

2.4 *Hierarchy Composition Rule*

Two Decomposition-Incorporation Theorems are given in (Hou, 2012).

Proposition 2.3. (Decomposition-Incorporation Theorem I) Suppose $D=\left(\overline{u}_{ij}\right)_{n\times m}$ and $B=\left(\beta_l\right)_{m\times 1}$, where $\beta_l>0$ and $\sum_{l=1}^{m}\beta_l=1$. If $\forall i,j\left(\overline{u}_{ij}>0\right)$, we have

$$r_j=\prod_{l=1}^{m}\left(\overline{u}_{jl}\right)^{\beta_l}$$
$$=\left(\prod_{l=1}^{t}\left(\overline{u}_{jl}\right)^{\frac{\beta_l}{\sum_{i=1}^{t}\beta_i}}\right)^{\sum_{i=1}^{t}\beta_i}\left(\prod_{l=t+1}^{m}\left(\overline{u}_{jl}\right)^{\frac{\beta_l}{\sum_{i=t+1}^{m}\beta_i}}\right)^{\sum_{i=t+1}^{m}\beta_i} \quad (5)$$
$$j=1,2,\ldots,n.$$

Proposition2.4.(Decomposition-Incorporation Theorem II) Suppose $D=\left(\overline{u}_{ij}\right)_{n\times m}$ and $B=\left(\beta_l\right)_{m\times 1}$, where $\beta_l>0$ and $\sum_{l=1}^{m}\beta_l=1$. If $\forall i,j\left(\overline{u}_{ij}\geq 0\right)$, we have

$$o_j=\sum_{l=1}^{m}\beta_l\left(\overline{u}_{jl}\right)$$
$$=\left(\sum_{i=1}^{t}\beta_i\right)\sum_{l=1}^{t}\frac{\beta_l}{\sum_{i=1}^{t}\beta_i}\left(\overline{u}_{jl}\right)+\left(\sum_{i=t+1}^{m}\beta_i\right)\sum_{l=t+1}^{m}\frac{\beta_l}{\sum_{i=t+1}^{m}\beta_i}\left(\overline{u}_{jl}\right), \quad (6)$$
$$j=1,2,\ldots,n.$$

The Decomposition-Incorporation Theorems are simple and their forms are not unique. However, they indicate when and how to decompose and incorporate a MCDA problem:

- If the criteria' weights w.r.t. the total goal are sum-normalized, then, the MCDA problem can be decomposed (one level to multi-level);
- If the weights of the same level sub-criteria dominated by the same immediate upper level criterion are sum-normalized, then, the decision problem can be incorporated (multi-level to one level).

Moreover, the Decomposition-Incorporation Theorem indicates how to aggregate the criteria weights between hierarchies:

- Let $\overline{\beta}_j^{(l+1)}$ denote the local weight of a sub-criterion in level $l+1$ w.r.t. its immediately preceding criterion/sub-criterion in level l (called the father-criterion);
- Let $\beta_j^{(l+1)}$ denote the global weight of the sub-criterion w.r.t. the total goal;
- Let $\beta_k^{(l)}$ denote the father-criterion's weight w.r.t. the total goal.

Then, the Hierarchy Composition Rule is

$$\beta_j^{(l+1)}=\beta_k^{(l)}\overline{\beta}_j^{(l+1)} \quad (7)$$

Remark 1 We should note that, recently, some other researchers have also done some valuable work of the PCM from the prospective of the abstract algebra as can be seen in (Barzilai, 1998; Cavallo and D'Apuzzo, 2009; Elsner and Van den Driessche, 2004). The min-normalization of the multiplicative PCM was suggested in Schoner, Wedley, & Choo (1993). The weighted geometric aggregation rule was supported by Barzilai (2001), Lootsma (1993), and many others. Using the geometric mean to derive weights from a multiplicative PCM was supported by Barzilai (1998), Crawford (1987), and Lootsma (1993), and also by many others. The function $\phi(x) = 0.5(1+\log_9 x)$ was first introduced in Fedrizzi (1990). The value 9 of the logarithm's basis in $\phi(x)$ was used to guarantee the compatibility with Saaty's scale $\{1/9,\ldots,1,\ldots,9\}$. Successive studies were also conducted in Chiclana et al. (2001) and Herrera et al. (2004).

3 Acceptable consistency criterion for PCMs

Two elements in a column of a PCM represent e.g. p_{24} and p_{34} the comparison results of object 2 and object 3 with respect to object 4. Thus, an acceptable consistency is defined as follows:

- For two rows of a PCM, if the relation ≤ holds elementwise, or, if the relation ≥ holds elementwise, then, these two rows are in acceptable consistency;
- If any two rows are in acceptable consistency, then, the PCM has acceptable consistency.

Based on this intuitive definition, we propose the following acceptable criterion.

Acceptable Consistency Criterion for multiplicative PCM A multiplicative pairwise comparison matrix $\mathbf{M} = (p_{ij})$ is of acceptable consistency if, and only if, the following condition is verified

$$(p_{ik} > p_{jk}) \to \forall l\,(p_{il} \geq p_{jl}). \qquad (8)$$

Clearly, we have

- If a PCM $\mathbf{M} = (p_{ij})$ is consistent, it must be acceptably consistent, since $p_{ij} = \omega_i / \omega_j$, where $\omega_i > 0$ and $\omega_j > 0$;
- It is not necessarily true that, when a PCM has acceptable consistency, it must be consistent.

For instance, we consider the matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 8 & 9 \\ 1/8 & 1 & 8 \\ 1/9 & 1/8 & 1 \end{pmatrix}.$$

It has the defined acceptable consistency but it is not a consistent one. Thus, the proposed acceptable criterion is less strict than the consistency condition, and hence we take it as the criterion for acceptable consistency. It is a criterion independent of the scale (for the scales used for PCM, see Triantaphyllou et al. (1994).

The above criterion can easily be extended to other kinds of PCM, for instance, to the fuzzy PCM.

Acceptable Consistency Criterion for fuzzy PCM A fuzzy pairwise comparison matrix $\mathbf{A} = (a_{ij})$ is of acceptable consistency if, and only if, the following condition is verified

$$(a_{ik} > a_{jk}) \to \forall l\,(a_{il} \geq a_{jl}). \qquad (9)$$

Remark 2 Under the isomorphism mapping $\phi(x) = 0.5(1+\log_\alpha x)$ with $\alpha > 1$, it can be easily proven that the indicator of inequality (9) is the counterpart of (8), since they can be written with semiring operations as

$$(a_{ik} \oplus_+ a_{jk} = a_{ik}) \wedge (a_{ik} \neq a_{jk}) \to \forall l\,(a_{il} \oplus_+ a_{jl} = a_{il})$$

and

$$(p_{ik} \oplus_\times p_{jk} = p_{ik}) \wedge (p_{ik} \neq p_{jk}) \to \forall l\,(p_{il} \oplus_\times p_{jl} = p_{il})$$

,

respectively.

4. Hierarchical model and steps

As already mentioned and discussed in above sections, the counterparts for criterion acceptable consistency, approach for deriving weights, hierarchical composition and aggregation rule are all included in our semiring isomorphic investigation results. Thus, we propose a Multiplicative Pairwise Comparison based Hierarchical Decision Model (MPCbHDM): (1) the criterion indicated by inequality (8) is used to check acceptable consistency; (2) the row's geometric mean method of Eq. (1) is used for deriving the local weights; (3) the Hierarchy Composition Rule of Eq.(7) is used for computing the sub-criterion's global weights; and (4) the weighted geometric mean method of Eq.(5) is used as the aggregation rule, where the alternative's local weights are min-normalized as indicated by Eq.(3). The MPCbHDM has the property of preserving rank. Moreover, it has counterparts in the fuzzy case.

For application, the steps of the MPCbHDM are elaborated as follows.

Step 1: Break down the decision problem into a hierarchy of decision elements (goal as the top level, criteria and sub-criteria as the middle levels, and alternatives as the terminal level);

Step 2: Establish the PCMs based on a ratio scale for the decision elements in each level of the hierarchy with respect to one decision element at a time in the immediate upper level.

Step 3: Determine whether or not the PCMs have acceptable consistency by indicator of inequality (8):

$$\left(p_{ik} > p_{jk}\right) \rightarrow \forall l \left(p_{il} \geq p_{jl}\right).$$

If not, go back to Step 2 and redo the pairwise comparisons.

Step 4: Derive the normalized local weight vectors from the PCMs using the row geometric mean method (Eq.(1)):

$$\omega_j = \left(\prod_{k=1}^{n} p_{jk}\right)^{\frac{1}{n}}. \quad (10)$$

If a local weight vector is of the criteria(sub-criteria) in a same level with respect to a specific decision element in the immediate upper level, it is then to be sum-normalized; if it is of the alternatives with respect to a terminal criterion, it is then to be min-normalized as indicated by Eq.(3):

$$\bar{\omega}_i^{\min} = \omega_i \big/ \min_k\{\omega_k\}.$$

Step 5: Compute the terminal sub-criteria (criteria) weights with respect to the total goal (using the Hierarchy Composition Rule of Eq.(7)):

$$\beta_j^{(l+1)} = \beta_k^{(l)} \bar{\beta}_j^{(l+1)}.$$

Step 6: To get an overall priority for each alternative, synthesize the alternative's local weights using the weighted geometric mean aggregation rule Eq.(5):

$$r_j = \prod_{l=1}^{m} \bar{u}_{jl}^{\beta_l}.$$

where, $\bar{u}_{jl}$ are the alternative's local weight (has been min-normalized in Step 4) w.r.t. the terminal sub-criterion, and, β_l is the terminal sub-criterion's weight w.r.t. the total goal.

5. An application example

We have developed a software tool which allows the MPCbHDM users to obtain the result conveniently with a computer.

For illustrating the application process, we take a conference site selection problem as an example. The total goal is to evaluate 3 alternatives, A_1, A_2 and A_3 with respect to multiple criteria. The criterion hierarchy is described by Figure 1, where the criterion's meaning is given by Table 1. The involved PCMs are constructed by the steering committee of the conference on the 1 to 5 ratio scale, as shown in Table 2 and Table 3

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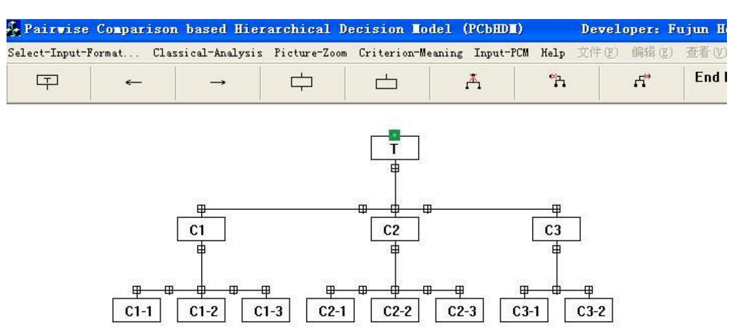

Figure 1: The criterion hierarchy

- Since all the involved PCMs are of satisfied consistency by the indicator of inequality (8), the local weighs can be derived as provided in Table 2 and Table 3;
- By using the Hierarchy Composition Rule of equation (7), we obtain the terminal criteria' global weights with respect to the total goal as: 0.179729, 0.098909, 0.054431, 0.324377, 0.189696, 0.055468, 0.048695, 0.048695;
- From the aggregation rule of equation (5), we obtain the alternatives' overall priorities as: 3.964602, 3.021387, 1.155347.

Thus, the ranking of the considered alternatives is: $A_1 \succ A_2 \succ A_3$.

Table 1. Meaning of the criteria

Criterion	Meaning
C1	Traffic convenience
C2	Hotel
C3	Historical culture and natural scenery
C1-1	Aircraft flight
C1-2	High speed railway
C1-3	City bus
C2-1	Hotel accommodation
C2-2	Conference facilities
C2-3	Room price
C3-1	Historical culture
C3-2	Natural scenery

Table 2: PCMs of criterion w.r.t. criterion and local weights

Pairwise comparison Matrix (PCM)	Local weights
C1,C2,and C3 w.r.t. the total goal: $\begin{pmatrix} 1 & 1/2 & 4 \\ 2 & 1 & 5 \\ 1/4 & 1/5 & 1 \end{pmatrix}$	The PCM HAS acceptable consistency (by indicator of inequality (8)). Local weights of the sub-criterion (by Eq.(1) and sum-normalized): 0.333069, 0.569541, 0.097390
C1-1,C1-2 and C1-3 w.r.t. C1: $\begin{pmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 2 \\ 1/3 & 1/2 & 1 \end{pmatrix}$	The PCM HAS acceptable consistency (by indicator of inequality (8)). Local weights of the sub-criterion (by Eq.(1) and sum-normalized): 0.539615, 0.296961, 0.163424
C2-1,C2-2 and C2-3 w.r.t. C2: $\begin{pmatrix} 1 & 2 & 5 \\ 1/2 & 1 & 4 \\ 1/5 & 1/4 & 1 \end{pmatrix}$	The PCM HAS acceptable consistency (by indicator of inequality (8)). Local weights of the sub-criterion (by Eq.(1) and sum-normalized): 0.569541, 0.333069, 0.097390
C3-1 and C3-2 w.r.t. C3: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	The PCM HAS acceptable consistency (by indicator of inequality (8)). Local weights of the sub-criterion (by Eq.(1) and sum-normalized): 0.500000, 0.500000

6. Conclusion

From one level MCDM to multi-level hierarchy decision model, the Decomposition- Incorporation Theorem assures its possibility and practicability. For pairwise comparison based decision making, the isomorphic counterparts of the multiplicative case and the fuzzy case indicate what we should do when we use hierarchical decision model to get overall priorities for alternatives. Thus, we elaborated a practicable procedure of steps for application based on our study. To facilitate our approach, a software tool with graphics user interface was developed which is available to interested readers by email.

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Table 3: PCMs of alternative w.r.t. terminal criterion and local weights

Pairwise comparison Matrix (PCM)	Local weights
A_1,A_2 and A_3 w.r.t. C1-1: $\begin{pmatrix} 1 & 2 & 5 \\ 1/2 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{pmatrix}$	The PCM HAS acceptable consistency (by indicator of inequality (8)). Local weights of the sub-criterion (by Eq.(1) and min-normalized): 5.313295, 2.823108, 1.000000
A_1,A_2 and A_3 w.r.t. C1-2: $\begin{pmatrix} 1 & 1/2 & 4 \\ 2 & 1 & 5 \\ 1/4 & 1/5 & 1 \end{pmatrix}$	The PCM HAS acceptable consistency (by indicator of inequality (8)). Local weights of the sub-criterion (by Eq.(1) and min-normalized): 3.419953, 5.848039, 1.000000
A_1,A_2 and A_3 w.r.t. C1-3: $\begin{pmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 2 \\ 1/3 & 1/2 & 1 \end{pmatrix}$	The PCM HAS acceptable consistency (by indicator of inequality (8)). Local weights of the sub-criterion (by Eq.(1) and min-normalized): 3.301929, 1.817121, 1.000000
A_1,A_2 and A_3 w.r.t. C2-1: $\begin{pmatrix} 1 & 2 & 5 \\ 1/2 & 1 & 4 \\ 1/5 & 1/4 & 1 \end{pmatrix}$	The PCM HAS acceptable consistency (by indicator of inequality (8)). Local weights of the sub-criterion (by Eq.(1) and min-normalized): 5.848039, 3.419953, 1.000000
A_1,A_2 and A_3 w.r.t. C2-2: $\begin{pmatrix} 1 & 2 & 4 \\ 1/2 & 1 & 3 \\ 1/4 & 1/3 & 1 \end{pmatrix}$	The PCM HAS acceptable consistency (by indicator of inequality (8)). Local weights of the sub-criterion (by Eq.(1) and min-normalized): 4.578859, 2.620742, 1.000000
A_1,A_2 and A_3 w.r.t. C2-3: $\begin{pmatrix} 1 & 1/2 & 1/4 \\ 2 & 1 & 2 \\ 4 & 1/2 & 1 \end{pmatrix}$	The PCM HAS acceptable consistency (by indicator of inequality (8)). Local weights of the sub-criterion (by Eq.(1) and min-normalized): 1.000000, 2.000000, 4.000000
A_1,A_2 and A_3 w.r.t. C3-1: $\begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 3 \\ 1/2 & 1/3 & 1 \end{pmatrix}$	The PCM HAS acceptable consistency (by indicator of inequality (8)). Local weights of the sub-criterion (by Eq.(1) and min-normalized): 1.817121, 3.301929, 1.000000
A_1,A_2 and A_3 w.r.t. C3-2: $\begin{pmatrix} 1 & 1/2 & 1/4 \\ 2 & 1 & 1/2 \\ 4 & 2 & 1 \end{pmatrix}$	The PCM HAS acceptable consistency (by indicator of inequality (8)). Local weights of the sub-criterion (by Eq.(1) and min-normalized): 1.000000, 2.000000, 4.000000

Appendix

A.1 Two semirings and isomorphism mapping

The following two algebra systems are semirings Hou (2011):

- $\mathrm{K}_{\times}=\left([0,+\infty),\oplus_{\times},\otimes_{\times},\varepsilon_{\times},e_{\times}\right)$ with $\varepsilon_{\times}=0$ and $e_{\times}=1$, where $x\oplus_{\times}y=\max\{x,y\}$ and $x\otimes_{\times}y=xy$ for any $x,y\in[0,+\infty)$;
- $\mathrm{K}_{+}=\left([-\infty,+\infty),\oplus_{+},\otimes_{+},\varepsilon_{+},e_{+}\right)$ with $\varepsilon_{+}=-\infty$ and $e_{+}=0.5$, where $x\oplus_{+}y=\max\{x,y\}$ and $x\otimes_{+}y=x+y-0.5$ for any $x,y\in[-\infty,+\infty)$;

The above two special semirings are introduced for dealing with the multiplicative PCM and the fuzzy PCM (fuzzy preference relations), respectively. The function

$\phi(x)=0.5\left(1+\log_{\alpha}x\right)$, where $\alpha\in(1,+\infty)$, is an isomorphism mapping between $\mathrm{K}_{\times}$ and K_{+} if we set $\phi(0)=-\infty$ and $\phi^{-1}(-\infty)=0$.

Proposition A.1. (Hou, 2012), The isomorphic counterpart of the weighted geometric mean (WGM) in the semiring $\mathrm{K}_{\times}$ is the weighted arithmetic mean (WAM) in semiring K_{+}. Namely,

$$\phi\Big(\prod_{k=1}^{m}g_k^{h_k}\Big)=\sum_{k=1}^{m}\big(h_k\phi(g_k)\big),$$

where $\forall j\left(g_j\in(0,+\infty),h_j\in(0,1)\right)$ and

$$\sum_{k=1}^{m}h_k=1.$$

A.2 Some counterparts for PCMs

The semiring-sense counterpart results were given by Hou (2011, 2012) (some are provided by Table A.1).

Table A.1: Some counterparts of the multiplicative case and the fuzzy case

Item	Multiplicative case	Fuzzy case
PCM definition	$\forall j,k\left(p_{jk}\otimes_{\times}p_{kj}=e_{\times}\right)$	$\forall j,k\left(a_{jk}\otimes_{+}a_{kj}=e_{+}\right)$
Consistency definition	$\forall j,k,l\left(p_{jl}\otimes_{\times}p_{lk}=p_{jk}\right)$	$\forall j,k,l\left(a_{jl}\otimes_{+}a_{lk}=a_{jk}\right)$
Priorities corresponding to consistent PCMs	$p_{jk}\otimes_{\times}\omega_{k}=\omega_{j}$	$a_{jk}\otimes_{+}v_{k}=v_{j}$
A necessary condition for consistency	$M\otimes_{\times}M=M$	$A\otimes_{+}A=A$
Necessary-sufficient condition for consistency	$M\otimes_{\times}W=W$	$A\otimes_{+}V=V$
Method for deriving weights	$\omega_{j}=\left(\prod_{k=1}^{n}p_{jk}\right)^{1/n}$	$v_{j}=\frac{1}{n}\left(\sum_{k=1}^{n}a_{jk}\right)$
Rank preserved aggregation rule	$r_{j}=\prod_{l=1}^{m}\overline{u}_{jl}^{\beta_{l}}$	$o_{j}=\sum_{k=1}^{m}\beta_{k}\overline{u}_{jk}$

In Table A.1, the necessary-sufficient condition for consistency takes the form of $f(x) = x$, thus it is called a fixed point equation (Hou, 2011). The aggregation rules presented in Table A.1 have the property of rank preservation. It was proven that when alternatives' local weights with respect to the terminal criterion are min-normalized, these two aggregation rules preserve the rank (Hou, 2012).

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