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نمذجة كسرية لمعادلة ب ب م- بيرجر (BBM-Burger) باستخدام طريقة
تحويل هوموتوبي (Homotopy) الجديدة

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المخلص:

ان الهدف من هذه الدراسة هو تقديم طريقة تحليلية جديدة وهي طريقة تحويل تحليل هوموتوبي الكسرية (FHATM) لحل السلسلة لمعادلة ب ب م- بيرجر (BBM Burger) الزمنية الكسرية. ان طريقة تحويل تحليل هوموتوبي هي تعديل مبنكر في تحويل لابلاس (LTA) لمعادلة تفاضلية جزئية كسرية لاختية في ديناميكا الموائع والتي تجعل الحسابات اكثر بساطة. ان الطريقة المقترحة تعمل على ايجاد حلول للمسائل الغير خطية بدون تفرد او افتراضات مقيدة وكما انها تعمل على تجنب اخطاء التقريب. ان الحلول العددية التي تم الحصول عليها من الطريقة المقترحة تدل على ان النهج سهل التنفيذ وحسابيا جذاب.



ORIGINAL ARTICLE

Fractional modelling for BBM-Burger equation by using new homotopy analysis transform method



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Abstract The purpose of this study is to introduce a new analytical method namely, fractional homotopy analysis transform method (FHATM) for series solution of the time fractional BBM-Burger equation. The homotopy analysis transform method is an innovative adjustment in Laplace transform algorithm (LTA) for nonlinear fractional partial differential equation in fluid dynamics and makes the calculation much simpler. The proposed scheme finds the solutions of nonlinear problems without any discretization, restrictive assumptions and avoids the rounding off errors. The numerical solutions obtained by the proposed method indicate that the approach is easy to implement and computationally very attractive.

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1. Introduction

The fractional calculus has a long history, starting from 30 September 1695 when the derivative of order $\alpha = 1/2$ was described by Leibniz (Oldham and Spanier, 1974). Fractional order ordinary differential equations, as generalizations of classical integer order ordinary differential equations, are increasingly used to model problems in fluid flow, mechanics, viscoelasticity, biology, physics and engineering, and other applications (Podlubny, 1999). Fractional derivatives provide an excellent instrument for the description of memory and

hereditary properties of various materials and processes. Half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models (Oldham and Spanier, 1974; Miller and Ross, 1993; Samko et al., 1993; Hilfer, 2000; Podlubny, 1999; Kilbas et al., 2006).

In this paper, the homotopy analysis transform method (HATM) basically illustrates how the Laplace transform can be used to find the approximate solutions of the time fractional BBM-Burger equation by manipulating the homotopy analysis method. The proposed method is coupling of the homotopy analysis method and Laplace transform method. The main advantage of this proposed method is its capability of combining two powerful methods for obtaining the approximate solution of time fractional BBM-Burger equation. Homotopy analysis method (HAM) was first proposed and applied by Liao (1992, 1997, 2003, 2004) based on homotopy, a fundamental concept in topology and differential geometry. The HAM has

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been successfully applied by many researchers for solving linear and non-linear partial differential equations (Abbasbandy, 2008, 2010, 2011, 2013; Jafari et al., 2010; Khan et al. 2012; Li et al., 2013; Vishal et al., 2012; Zhang et al., 2011). In recent years, many researchers have paid attention in obtaining solutions to linear and nonlinear differential, and integral equations by various methods by combining the Laplace transform method. Among these we may mention the following: the Laplace decomposition methods (Jafari et al., 2013; Khan et al., 2012; Khan et al., 2013; Wazwaz, 2010), homotopy perturbation transform method (Kumar et al., 2012a, 2012b, 2013a, 2013b; Singh et al., 2013). Recently, Khan et al. (2012) have applied to obtain the solutions of the Blasius flow equation on a semi-infinite domain by coupling of homotopy analysis and Laplace transform method. Recently, many researchers (Arife et al., 2013; Kumar et al., 2013; Zurigat, 2011) have solved fractional differential equation by using modified homotopy analysis method with Laplace transform method. The different type solutions of the fractional BBM-Burger equation have been discussed by Fakhari et al. (2007), Song and Zhang (2009) by using homotopy analysis method.

This paper is committed to the study of time fractional BBM-Burger equation by using new fractional homotopy analysis transform method. The BBM-Burger equation can be written in time fractional operator form as

$$D_t^\alpha u - u_{xxt} + u_x + \left(\frac{u^2}{2}\right)_x = 0, \quad t > 0, \quad 0 < \alpha \leq 1, \quad (1.1)$$

with initial condition $u_0(x, t) = \sec h^2\left(\frac{x}{4}\right)$ and α is a parameter describing the order of the time fractional derivative and lie in the interval $(0, 1]$. We remark that the exact travelling wave solution $u(x, t) = \sec h^2\left(\frac{x}{4} - \frac{t}{4}\right)$ to the above initial value problem is given by (Fakhari et al., 2007).

Definition 1.1. The Laplace transform of continuous (or an almost piecewise continuous) function $f(t)$ in $[0, \infty)$ is defined as

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t) dt. \quad (1.2)$$

where s is a real or complex number.

Definition 1.2. The Laplace transform $L[f(t)]$ of the Riemann–Liouville fractional integral is defined as (Podlubny, 1999):

$$L[I_t^\alpha f(t)] = s^{-\alpha} F(s). \quad (1.3)$$

Definition 1.3. The Laplace transform $L[f(t)]$ of the Caputo fractional derivative is defined as (Podlubny, 1999)

$$L[D_x^\alpha u(x, t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{(\alpha-k-1)} u_x^{(k)}(0, t), \quad n-1 < \alpha \leq n. \quad (1.4)$$

2. Basic idea of newly fractional homotopy analysis transform method (FHATM)

To illustrate the basic idea of the FHATM for the fractional partial differential equation, we consider the following fractional partial differential equation as:

$$D_t^\alpha u(x, t) + R[x]u(x, t) + N[x]u(x, t) = g(x, t), \quad t > 0, x \in R, n-1 < \alpha \leq n, \quad (2.1)$$

where $D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$, $R[x]$ is the linear operator in x , $N[x]$ is the general nonlinear operator in x , and $g(x, t)$ are continuous functions. For simplicity we ignore all initial and boundary conditions, which can be treated in a similar way. Now the methodology consists of applying the Laplace transform first on both sides of Eq. (2.1), we get

$$L[D_t^\alpha u(x, t)] + L[R[x]u(x, t) + N[x]u(x, t)] = L[g(x, t)]. \quad (2.2)$$

Now, using the differentiation property of the Laplace transform, we have

$$L[u(x, t)] - \frac{1}{s^{\alpha}} \sum_{k=0}^{n-1} s^{(\alpha-k-1)} u^k(x, 0) + \frac{1}{s^{\alpha}} L[R[x]u(x, t) + N[x]u(x, t) - g(x, t)] = 0. \quad (2.3)$$

We define the nonlinear operator

$$N[\phi(r, t; q)] = L[\phi(x, t; q)] - \frac{1}{s^{\alpha}} \sum_{k=0}^{n-1} s^{(\alpha-k-1)} u^k(x, 0) + \frac{1}{s^{\alpha}} L(R[x]u(x, t) + N[x]u(x, t) - g(x, t)), \quad (2.4)$$

where $q \in [0, 1]$ be an embedding parameter and $\phi(x, t; q)$ is the real function of x, t and q . By means of generalising the traditional homotopy methods, Liao (1992, 1997, 2003, 2004) constructed the zero order deformation equation

$$(1 - q)L[\phi(x, t; q) - u_0(x, t)] = \hbar q H(x, t) N[\phi(x, t; q)], \quad (2.5)$$

where \hbar is a nonzero auxiliary parameter, $H(x, t) \neq 0$ an auxiliary function, $u_0(x, t)$ is an initial guess of $u(x, t)$ and $\phi(x, t; q)$ is an unknown function. It is important that one has great freedom to choose auxiliary thing in FHATM. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(x, t; 0) = u_0(x, t), \quad \phi(x, t; 1) = u(x, t), \quad (2.6)$$

respectively. Thus, as p increases from 0 to 1, the solution varies from the initial guess $u_0(x, t)$ to the solution $u(x, t)$. Expanding $\phi(x, t; q)$ in Taylor's series with respect to q , we have

$$\phi(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} q^m u_m(x, t), \quad (2.7)$$

where

$$u_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \phi(x, t; q)}{\partial q^m} \right|_{q=0}. \quad (2.8)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter \hbar , and the auxiliary function are properly chosen, the series (2.7) converges at $q = 1$, we have

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t), \quad (2.9)$$

which must be one of the solutions of the original nonlinear equations.

Defines the vectors

$$\vec{u}_n = \{u_0(x, t), u_1(x, t), u_2(x, t), \dots, u_n(x, t)\}. \quad (2.10)$$

Differentiating Eq. (2.5) m time with respect to embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we obtain the m th order deformation equation

$$L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar q H(x, t) R_m(\vec{u}_{m-1}, x, t). \quad (2.11)$$

Operating the inverse Laplace transform on both sides, we get

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + \hbar q L^{-1} [H(x, t) R_m(\vec{u}_{m-1}, x, t)], \quad (2.12)$$

where

$$R_m(\vec{u}_{m-1}, x, t) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \phi(x, t; q)}{\partial q^{m-1}} \right|_{q=0}, \quad (2.13)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

In this way, it is easy to obtain $u_m(x, t)$ for $m \geq 1$, at M th order, we have

$$u(x, t) = \sum_{m=0}^M u_m(x, t), \quad (2.14)$$

when $M \rightarrow \infty$ we get an accurate approximation of the original Eq. (2.1).

3. Solution of the given problem by a newly proposed method

We first consider the following time-fractional BBM-Burger equation as (Fakhari et al., 2007).

$$D_t^\alpha u - u_{xx} + u_x + \left(\frac{u^2}{2}\right)_x = 0, \quad t > 0, \quad 0 < \alpha \leq 1, \quad (3.1)$$

with initial condition

$$u(x, 0) = \sec h^2\left(\frac{x}{4}\right). \quad (3.2)$$

Applying the Laplace transform on both sides in Eq. (3.1) and after using the differentiation property of Laplace transform for fractional derivative, we get

$$s^\alpha L[u(x, t)] - s^{\alpha-1} u(x, 0) + L\left[u_x - u_{xx} + \left(\frac{u^2}{2}\right)_x\right] = 0. \quad (3.3)$$

On simplifying

$$L[u(x, t)] - \frac{1}{s} \sec h^2\left(\frac{x}{4}\right) + s^{-\alpha} L\left[u_x - u_{xx} + \left(\frac{u^2}{2}\right)_x\right] = 0. \quad (3.4)$$

We choose the linear operator as

$$\mathcal{L}[\phi(x, t; q)] = L[\phi(x, t; q)], \quad (3.5)$$

with property $\mathcal{L}[c] = 0$ where c is constant.

We now define a nonlinear operator as

$$N[\phi(x, t; q)] = L[\phi(x, t; q)] - \frac{1}{s} \sec h^2\left(\frac{x}{4}\right) + s^{-\alpha} L\left[\phi_x - \phi_{xx} + \left(\frac{\phi^2}{2}\right)_x\right]. \quad (3.6)$$

Using the above definition, with assumption $H(x, t) = 1$, we construct the zeroth order deformation equation

$$(1 - q)\mathcal{L}[\phi(x, t; q) - u_0(x, t)] = q\hbar N[\phi(x, t; q)]. \quad (3.7)$$

Obviously, when $q = 0$ and $q = 1$,

$$\phi(x, t; 0) = u_0(x, t), \quad \phi(x, t; 1) = u(x, t). \quad (3.8)$$

Thus, we obtain the m th order deformation equation

$$L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar R_m(\vec{u}_{m-1}, x, t). \quad (3.9)$$

Operating the inverse Laplace transform on both sides in Eq. (3.9), we get

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + \hbar q L^{-1}[R_m(\vec{u}_{m-1}, x, t)], \quad (3.10)$$

where

$$R_m(\vec{u}_{m-1}, x, t) = L[u_{m-1}(x, t)] - \frac{1}{s} \sec h^2\left(\frac{x}{4}\right) + s^{-\alpha} L\left[(u_{m-1})_x - (u_{m-1})_{xx} + \sum_{k=0}^{m-1} u_{m-1-k}(u_k)_x\right]. \quad (3.11)$$

Now the solution of m th order deformation Eq. (3.9)

$$u_m(x, t) = (\chi_m + \hbar)u_{m-1} - \hbar(1 - \chi_m) \sec h^2\left(\frac{x}{4}\right) + \hbar L^{-1}(s^{-\alpha} L[(u_{m-1})_x - (u_{m-1})_{xx} + \sum_{k=0}^{m-1} u_{m-1-k}(u_k)_x]). \quad (3.12)$$

We start with initial condition $u_0(x, t) = u(x, 0) = \sec h^2\left(\frac{x}{4}\right)$, and the iterative scheme (3.12), we obtain the various iterates

$$u_1(x, t) = -\frac{\hbar t^\alpha (3 + \cosh\left(\frac{x}{4}\right)) \sec h^4\left(\frac{x}{4}\right) \tan\left(\frac{x}{4}\right)}{4\Gamma(\alpha + 1)},$$

$$u_2(x, t) = \frac{\hbar^2 t^{2\alpha} (-104 + 23 \cosh\left(\frac{x}{4}\right) + 16 \cosh x + \cosh\left(\frac{3x}{4}\right)) \sec h^8\left(\frac{x}{4}\right)}{128\Gamma(2\alpha + 1)} - \frac{\hbar(1 + \hbar)t^\alpha (3 + \cosh\left(\frac{x}{4}\right)) \sec h^4\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right)}{4\Gamma(\alpha + 1)} + \frac{\hbar^2 \alpha^{2\alpha-1} \Gamma(\alpha) \sec h^7\left(\frac{x}{4}\right) (\sinh\left(\frac{5x}{4}\right) + 7 \sinh\left(\frac{3x}{4}\right) - 14 \sinh\left(\frac{x}{4}\right))}{128\Gamma(2\alpha)\Gamma(\alpha + 1)}.$$

Proceeding in this manner, the rest of the components $u_n(x, t)$ for $n \geq 2$ can be completely obtained and the series solutions are thus entirely determined.

Finally, we have

$$u(x, t) = u_0(x, t) + \sum_{m=1}^5 u_m(x, t). \quad (3.13)$$

However, mostly; the results given by the Adomian decomposition method and homotopy perturbation transform method converge to the corresponding numerical solutions in a rather small region. But, different from those two methods, the homotopy analysis transform method provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameter \hbar . So the valid region for \hbar where the series converges is the horizontal segment of each \hbar curve. When we choose $\alpha = 1$ then clearly, we can conclude that the obtained solution $\sum_{m=0}^{\infty} u_m(x, t)$ converges to the exact solution $u(x, t) = \sec h^2\left(\frac{x}{4} - \frac{t}{4}\right)$, which is an exact solution of the standard BBM-Burger equation.

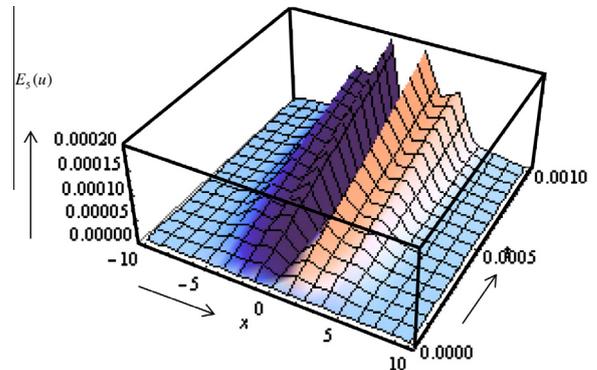


Figure 1 Absolute error $E_5(u)$ at $\alpha = 1$, $\hbar = -1$.

Table 1 Comparison study between the exact solution and 5th term approximations by FHATM.

x	t	Exact solution	Apr. solution at $\alpha = 1, h = -1$	$E_5(u)$ at $\alpha = 0.9, h = -1$	$E_5(u)$ at $\alpha = 1, h = -1$
20	0.01	0.0001825	0.0001828	1.60588×10^{-6}	3.03359×10^{-7}
15	0.01	0.0022210	0.0022247	1.95707×10^{-5}	3.7165×10^{-6}
10	0.01	0.0267237	0.0267690	2.13530×10^{-4}	4.52894×10^{-5}
20	0.001	0.0001817	0.0001817	2.8241×10^{-7}	3.01818×10^{-8}
15	0.001	0.0022110	0.0022114	3.43703×10^{-6}	3.69715×10^{-7}
10	0.001	0.0266053	0.0260985	3.39917×10^{-5}	4.50091×10^{-6}

4. Numerical result and discussion

The simplicity and accuracy of the proposed method are illustrated by computing the absolute errors $E_5(x, t) = |u(x, t) - \tilde{u}_5(x, t)|$ where $u(x, t)$ are the exact solutions and $\tilde{u}_5(x, t)$ are approximate solutions of (1.1) obtained by truncating the respective solution series (3.13) at level $m = 5$. Fig. 1 represents the absolute error which show our approximate solution converges to the exact solution very rapidly. From Fig. 1 of absolute error, it is seen that our approximate solutions obtained by fractional homotopy analysis transform method converges very rapidly to the exact solutions in only 5th order approximations. It achieves a high level of accuracy. The accuracy of the result can be improved by introducing more terms of the approximate solutions.

From Table 1, it is observed that the values of the approximate solution at different grid points obtained by the proposed method are close to the values of the exact solution with high accuracy at the level $m = 5$. It can also be noted that the accuracy increases as the value of n increases.

5. Concluding remarks

In this work, the authors have proposed a very effective method called the fractional homotopy analysis transform method (FHATM) for solution of time fractional BBM-Burger equation. The proposed iterative scheme finds the solution without any discretization, linearisation or restrictive assumptions. It may be concluded that FHATM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems.

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