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المقارنة بين طريقة التقارب الامثل الهوموتوبى و طريقة الاضطراب الهوموتوبى للمعادلة اللاخطية المعقدة التركيب

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الملخص:

في هذا البحث، تم توظيف طريقة للتحليل التقريبي وهي طريقة التقارب الامثل الهوموتوبي (**OHAM**). وذلك للتحقيق في التدفق البسيط (على شكل فيلم رقيق) لسائل من الرتبة الثالثة للاسفل على سطح مائل. وقد توصلت الطريقة المستخدمة الى حل دقيق على عكس الطرق المستخدمة سابقا في هذا المجال ذات النتائج الخاطئة. لقد تمت مقارنة قيم مختلفة للمعلومات في مجال السرعة للطريقة المستخدمة مع بعض من الطرق العددية منها: طريقة رانج كوتا فيلبيرج من المرتبة الرابعة والخامسة و طريقة الاضطراب الهوموتوبى (**HPM**). أخيرا، تم التوصل الى ان كل قيم المعلمات بطريقة **OHAM** هي ذات توافق جيد مع القيم المتباينة المستخرجة بطريقة **HPM**.

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ORIGINAL ARTICLE

Comparison of optimal homotopy asymptotic method and homotopy perturbation method for strongly non-linear equation

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KEYWORDS

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Abstract In this paper, we employ an approximate analytical method, namely the optimal homotopy asymptotic method (OHAM), to investigate a thin film flow of a third grade fluid down an inclined plane and provided accurate solution unlike other erroneous results available in the literature. The variation of the velocity field for different parameters is compared with the numerical values obtained by the *Runge–Kutta Fehlberg fourth–fifth order* numerical method and with the homotopy perturbation method (HPM). Finally, it was found that for all values of parameters OHAM agrees well with the numerical disparate HPM.

1. Introduction

Most scientific phenomena are inherently nonlinear such as heat transfer, and many of them have no analytical solution. Therefore, many different methods have been established by researchers to overcome such nonlinear problems. These methods include the artificial parameter method by (He, 2006a,b), the variational iteration method by (He, 2000), the homotopy analysis method by (Liao, 2003), the homotopy perturbation method by (He, 2006c) and the optimal homotopy asymptotic method by (Marinca and Herisanu, 2008) among others. The homotopy perturbation method (HPM) provides an approximate analytical solution in a series form. HPM has been widely used by numerous researchers successfully for different physical systems such as, bifurcation, asymptotology, nonlinear wave equations, oscillators with discontinuities by (He, 2004ab, 2005a,b), reaction-duffision equation and heat radiation equation by (Ganji and Rajabi, 2006; Ganji and Sadighi, 2006) and MHD Jeffery–Hamel problem by (Moghimi et al., 2011).

Significant classes of fluids commonly used in industries are non-Newtonian fluids. The applications of these fluids arise in areas such as synthetic fibers, food stuffs, drilling oil and gas wells, extrusion of molten plastics and polymers among others. The related literature indicates that the third grade fluid has been investigated by many researchers for different geometries and with different techniques.

Here, we consider the steady uni-directional flow of an incompressible third-grade fluid down a uniform inclined plane. For the third grade fluid, the first four terms of Taylor

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series are using the stress rate of strain relation. The third grade fluid models are complicated due to a large number of physical parameters that have to be determined experimentally.

The steady flow of third grade fluid in a bounded domain with Dirichlet boundary conditions analyzed by Adriana et al., 2008. Bresch and Lemoine (1999) have shown the existence of the solutions for non-stationary third-grade fluids and used homogenous boundary condition for the global and local existence of the fluid velocity equation. Many researchers (Zhang and Li, 2005; Busuioc et al., 2008; Khan and Mahmood, 2012; Siddiqui et al., 2008; Hayat et al., 2008, 2009; Kumaran et al., 2012) have investigated thin film flow of the third grade fluid, in addition Hameed and Ellahi, 2011 studied thin film flow for MHD fluid on moving belt. Moreover, Elahi and Riaz, 2010; Ellahi et al., 2011; Ellahi, 2012 successfully provided the series solution for non-Newtonian MHD flow with variable viscosity in a third grade fluid and discussed heat transfer in porous cylinder.

The optimal homotopy asymptotic method is an approximate analytical tool that is simple and straightforward and does not require the existence of any small or large parameter as does the traditional perturbation method. The optimal homotopy asymptotic method (OHAM) has been successfully applied to a number of nonlinear problems arising in fluid mechanics and heat transfer by various researchers (Herisanu et al., 2008; Mabood et al., 2013a,b; Marinca and Herisanu, 2008, 2010a,b).

Mathematical modeling of non-Newtonian fluid flow gives rise to nonlinear differential equations. Many numerical and analytical techniques have been proposed by various researchers. An efficient approximate analytical solution will find enormous applications. In this paper, we have solved the governing nonlinear differential equation of the present problem using OHAM and compared with numerical and HPM. It is important to mention here that the approximate analytical and numerical solutions are in a good agreement but better than the results of Siddiqui et al., 2008.

This paper is organized as follows: First in Section 2, governing equations of the problem are presented. In Section 3 we described the basic principles of OHAM. The OHAM solution is given in Section 4. In Section 5, outlines of HPM are discussed with HPM solution. In Section 6, we analyzed the comparison of the solution using OHAM with the numerical method and existing solution of HPM. Section 7 is devoted for the concluding remarks.

2. Governing equation

The thin film flow of an incompressible third grade fluid down on an inclined plane with inclination $\alpha \neq 0$ is governed by the following nonlinear boundary value problem in a dimensionless form (Siddiqui et al., 2008).

$$\frac{d^2u}{dy^2} + 6\beta\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} + m = 0 \tag{1}$$

Subject to the boundary conditions:

$$u(0) = 0, \qquad \frac{du}{dy} = 0 \quad \text{at} \quad y = 1 \tag{2}$$

As $m = \dfrac{g\rho\sin\alpha}{\mu}$, $\beta = \dfrac{(\beta_2 + \beta_3)}{\mu}$

where u is the fluid velocity, ρ is the density, μ is the dynamic viscosity, β_2 and β_3 are the material constants of the third grade fluid, g is acceleration due to gravity.

3. Basic principles of OHAM

We review the basic principles of OHAM as expounded in Herisanu et al., 2008 and other researchers (Mabood et al., 2013a; Marinca and Herisanu, 2008).

(i) Consider the following differential equation:

$$A[v(x)] + a(x) = 0, \quad x \in \Omega \tag{3}$$

where Ω is problem domain, $A(v) = L(v) + N(v)$, where L, N are linear and nonlinear operator, $v(x)$ is an unknown function, $a(x)$ is a known function,

(ii) Construct an optimal homotopy equation as:

$$(1-p)[L(\phi(x;p) + a(x)] - H(p)[A(\phi(x;p) + a(x)] = 0 \tag{4}$$

where $0 \leqslant p \leqslant 1$ is an embedding parameter, $H(p) = \sum_{k=1}^{m} p^k C_k$ is auxiliary function on which the convergence of the solution is greatly dependent. The auxiliary function $H(p)$ also adjusts the convergence domain and controls the convergence region.

(iii) Expand $\phi(x;p,C_j)$ in Taylor's series about p, one has an approximate solution:

$$\phi(x;p,\, C_j) = v_0(x) + \sum_{k=1}^{\infty} v_k(x,\, C_j)p^k, \quad j = 1,\, 2,\, 3, \ldots \tag{5}$$

Many researchers have observed that the convergence of the series Eq. (5) depends upon C_j, $(j = 1, 2, \ldots, m)$, if it is convergent then, we obtain:

$$\tilde{v} = v_0(x) + \sum_{k=1}^{m} v_k(x; C_j) \tag{6}$$

(iv) Substituting Eq. (6) in Eq. (4), we have the following residual:

$$R(x; C_j) = L(\tilde{v}(x; C_j)) + a(x) + N(\tilde{v}(x; C_j)) \tag{7}$$

If $R(x; C_j) = 0$, then $\tilde{v}$ will be the exact solution. For nonlinear problems, generally this will not be the case. For determining C_j, $(j = 1, 2, \ldots, m)$, Galerkin's Method, Ritz Method or the method of least squares can be used.

(v) Finally, substitute these constants in Eq. (7) and one can get the approximate solution.

4. Solution of the problem via OHAM

According to the OHAM, applying Eq. (4) to Eq. (1):

$$(1-p)(u'' + m) - H(p, C_i)\{u'' + 6\beta u'^2 u'' + m\} = 0 \tag{8}$$

where primes denote differentiation with respect to y.

We consider u and $H(p, C_i)$ as the following:

$$\begin{cases} u = u_0 + pu_1 + p^2u_2 \\ H(p, C_i) = pC_1 + p^2C_2 \end{cases} \tag{9}$$

Using Eq. (9) in Eq. (8) and some simplification and rearrangements of the terms based on the powers of p, we obtain zeroth, first and second order problems:

Zeroth order problem:

$$u_0''(y) = -m \tag{10}$$

with boundary conditions:

$$u_0(0) = 0, \; u_0'(1) = 0 \tag{11}$$

Its solution is

$$u_0 = \frac{1}{2}(2my - my^2) \tag{12}$$

First order problem:

$$u_1''(y, C_1) = m + mC_1 + 6\beta C_1(u_0')^2u_0'' + (1 + C_1)u_0''(y) \tag{13}$$

with boundary conditions:

$$u_1(0) = 0, \; u_1'(1) = 0 \tag{14}$$

having solution

$$u_1(y, C_1) = \frac{1}{2}(4m^3\beta yC_1 - 6m^3\beta y^2C_1 + 4m^3\beta y^3C_1 - m^3\beta y^4C_1) \tag{15}$$

Second order problem:

$$u_2''(y, C_1, C_2) = mC_2 + C_2u_0'' + 6\beta C_2(u_0')^2u_0'' + 12\beta C_1u_0'u_1'u_0'' + 6\beta C_1(u_0')^2u_1'' + (1 + C_1)u_1'' \tag{16}$$

with boundary conditions

$$u_2(0) = 0, \; u_2'(1) = 0 \tag{17}$$

Its solution becomes

$$\begin{aligned} u_2(y, C_1, C_2) = \tfrac{1}{2}(&4m^3\beta yC_1 - 6m^3\beta y^2C_1 + 4m^3\beta y^3C_1 - m^3\beta y^4C_1 + 4m^3\beta yC_1^2 + 24m^5\beta^2yC_1^2 \\ &-6m^3\beta y^2C_1^2 - 60m^5\beta^2y^2C_1^2 + 4m^3\beta y^3C_1^2 + 80m^5\beta^2y^3C_1^2 - m^3\beta y^4C_1^2 - 60m^5\beta^2y^4C_1^2 \\ &+24m^5\beta^2y^5C_1^2 - 4m^5\beta^2y^6C_1^2 + 4m^3\beta yC_2 - 6m^3\beta y^2C_2 + 4m^3\beta y^3C_2 - m^3\beta y^4C_2) \end{aligned} \tag{18}$$

We obtain the three terms' solution using OHAM for $p = 1$

$$\tilde{u}(y, C_1, C_2) = u_0(y) + u_1(y, C_1) + u_2(y, C_1, C_2) \tag{19}$$

We use the method of least squares to obtain the unknown convergent constants C_1, C_2 in Eq. (19).

For the particular case if $\beta = 0.5$ and $m = 1$, we have $C_1 = -0.20888457, \; C_2 = -0.04214067$

5. Outlines of HPM

We review the basic idea of HPM (He, 2006c).

Consider the following differential equation and boundary condition:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{20}$$

with boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \tag{21}$$

where A, B, $f(r)$, Γ are a general differential, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively. Generally speaking, the operator A can be divided into a linear part L and a nonlinear part $N(u)$. So Eq. (20) can be written as:

$$L(u) + N(u) - f(r) = 0 \tag{22}$$

By the homotopy method, we construct a homotopy $v(r, p) : \Omega \times [0, 1] \to R$ which satisfies:

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0, 1], \; r \in \Omega \tag{23}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{24}$$

where $p \in [0, 1]$ is an embedding parameter, while u_0 is an initial approximation of Eq. (20) which satisfies the boundary conditions. Obviously, from Eqs. (23), (24) we obtain:

$$H(v, 0) = L(v) - L(u_0) = 0 \tag{25}$$

$$H(v, 1) = A(v) - f(r) = 0 \tag{26}$$

The changing process of p from zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology, it is called deformation, while $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. According to the HPM, we can use embedding parameter p as a "small parameter", and assume that the solutions of Eqs. (23) and (24) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2v_2 + \ldots \tag{27}$$

Setting $p = 1$ yields in the approximate solution of Eq. (27) to:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \tag{28}$$

Solution of Eq. (1) with boundary conditions (Eq. (2)) via the homotopy perturbation method can be seen in Siddiqui et al., 2008. The second order series solution is:

$$u(y) = m\left(y - \frac{y^2}{2}\right) + 6\beta m^3\left(\frac{y^4}{12} - \frac{y^3}{3} + \frac{y^2}{2} - \frac{y}{3}\right) + 36m^5\beta^2\left(\frac{y}{3} - \frac{5y^2}{6} + \frac{10y^3}{9} - \frac{5y^4}{6} + \frac{y^5}{3} - \frac{y^6}{18}\right) \tag{29}$$

Table 1 Comparison of OHAM with NM and HPM (Siddiqui et al., 2008) for $\beta = 1.4$, $m = 0.75$.

x	OHAM	HPM	NM	Error (HPM)	Error (OHAM)
0.0	0	0	0	0	0
0.1	0.049261	0.221091	0.0484625	1.726	7.9×10^{-4}
0.2	0.095452	0.360656	0.0936873	0.266	1.7×10^{-3}
0.3	0.138530	0.449731	0.1353969	0.314	3.1×10^{-3}
0.4	0.178066	0.508866	0.1732599	0.335	4.8×10^{-3}
0.5	0.213394	0.55069	0.2068777	0.343	6.5×10^{-3}
0.6	0.243738	0.582128	0.2357687	0.346	7.9×10^{-3}
0.7	0.268316	0.606284	0.2593566	0.346	8.9×10^{-3}
0.8	0.286429	0.623979	0.2769773	0.347	9.4×10^{-3}
0.9	0.297529	0.634942	0.2879372	0.347	9.5×10^{-3}
1.0	0.301269	0.638672	0.2916666	0.347	9.6×10^{-3}

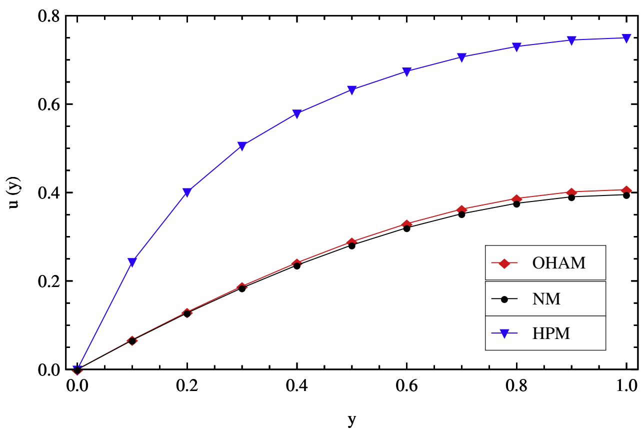

Figure 1 Comparison of velocity profile using OHAM, NM and HPM (Siddiqui et al., 2008) for $\beta = 0.5$, $m = 1$.

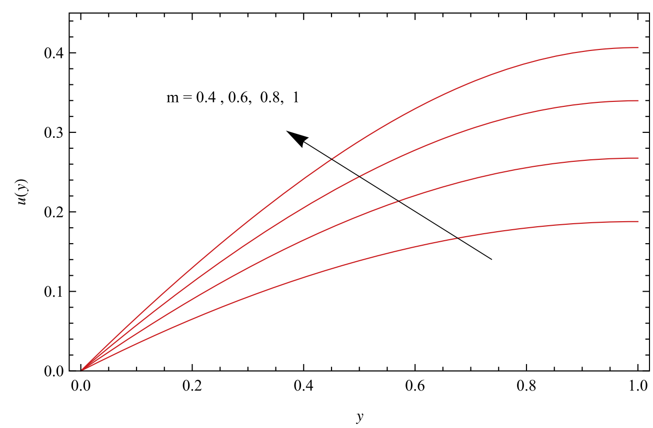

Figure 3 Effects on velocity profile for various values of m at $\beta = 0.5$.

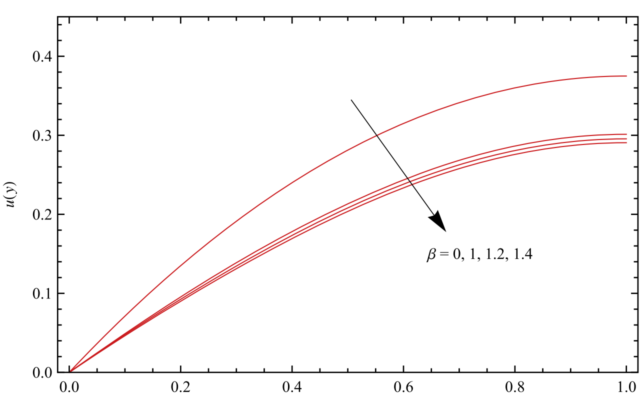

Figure 2 Effects on velocity profile for various values of β at $m = 0.75$.

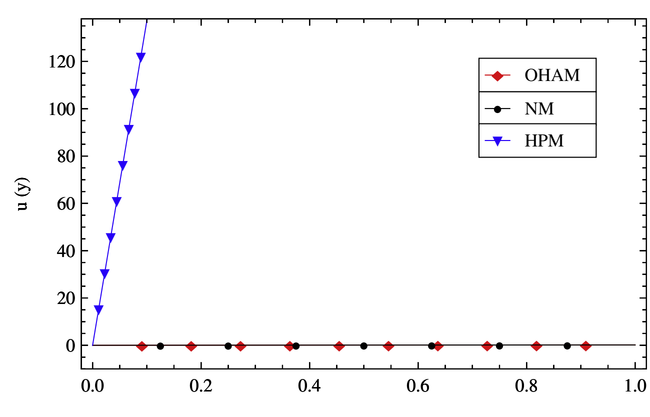

Figure 4 Comparison of velocity profile using OHAM, NM and HPM (Siddiqui et al., 2008) for $\beta = 25$, $m = 0.75$.

6. Results and discussion

This section presents the effects of controlling parameters on the velocity profile in the form of graphical and tabulated results. In order to validate the accuracy of our approximate solution via OHAM, we have presented a comparative study of OHAM solution with numerical and existing HPM solutions. The numerical results will be denoted by NM and HPM results by HPM. The numerical results are from the *Runge–Kutta Fehlberg fourth–fifth order* method and HPM results are from Siddiqui et al., 2008. Table 1 shows the comparison of our present OHAM results with NM and HPM for $\beta = 1.4$, $m = 0.75$ and the absolute errors. It is noteworthy to mention

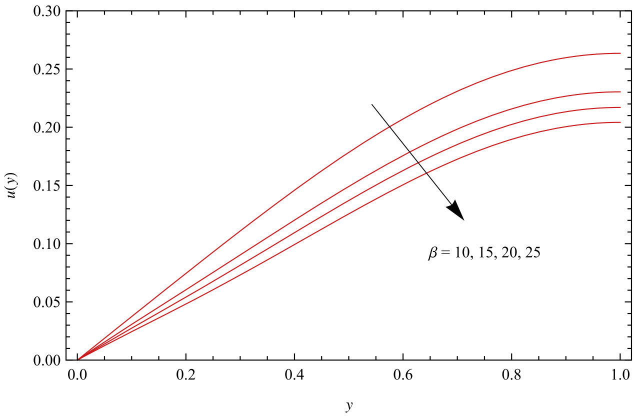

Figure 5 Effects on velocity profile for larger values of β at $m = 0.75$.

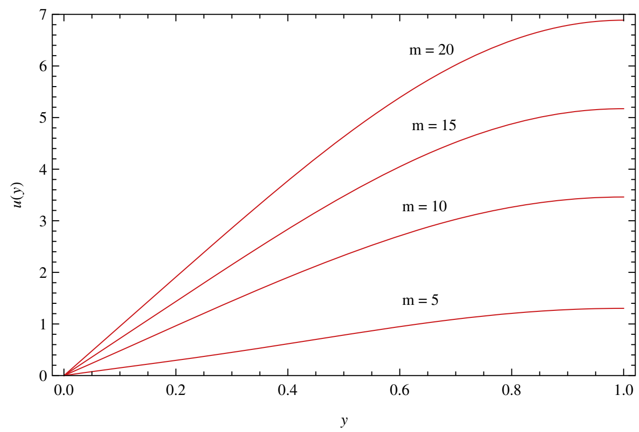

Figure 6 Effects on velocity profile for larger values of m at $\beta = 0.5$.

here that OHAM's lowest error is good as compared to HPM. The efficiency of OHAM can be concluded from Fig. 1 in which we compared the solution using OHAM with NM and HPM for particular values of the controlling parameters.

Fig. 2 illustrates the velocity profile for different values of the controlling parameters. For increasing values of parameter β and for the fixed value of m, a decrease in the velocity profile is observed, but (for the same values of the parameters β and m) the solution described in Fig. 1(a) of HPM is relatively opposite which is invalid and this was also noted by Hayat et al., 2008. Fig. 3 depicts that for increasing values of m keeping fixed value of β will cause the velocity profile to also increase. This is an agreement (in terms of velocity profile behavior) with the corresponding results for HPM shown in Fig. 1(b) of Siddiqui et al., 2008. However, the values of velocity profile of Fig. 3 obtained via OHAM are much closer to the numerical values as compared to HPM solution in Fig. 1(b) of Siddiqui et al., 2008.

It is important to note that, for the large values of fluid parameters β and m the solution of Siddiqui et al., 2008 is not correct. This was also pointed out by Hayat et al., 2008 and is shown in Fig. 4. But unfortunately, the velocity profile displayed in Figs. 1 and 2 of Hayat et al., 2008 is also not correct as pointed out by Kumaran et al., 2012 and have provided the correct version of Figs. 1 and 2 of Hayat et al., 2008. The approximate analytical solution via OHAM for the larger values of fluid parameters β and m can be seen from Figs. 5 and 6, confirming the strength of OHAM.

7. Concluding remarks

In this paper, we have studied a thin film flow of third grade fluid down an inclined plane. Both approximate analytical and numerical results are obtained for this nonlinear problem. The results are sketched and discussed for the fluid parameter β and for constant m. It is found that optimal homotopy asymptotic method (OHAM) results are much better than HPM results. For large values of non-Newtonian parameters HPM solution is invalid whereas OHAM solution is convincing. Finally, we conclude that OHAM provides a simple and easy way to control and adjust the convergence region for strong nonlinearity and is applicable to highly nonlinear fluid problems.

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