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حل معادلة من نوع $x^d = \beta$ في الزمرة المتناوبة لكل $n \in \theta$ و $\beta \in H_n \cap C^\alpha$

Shuker Mahmood[1,2,*], Andrew Rajah[2]

[1]Department of Mathematics, College of Science, University of Basra, Basra, Iraq
[2]School of Mathematical Sciences, Universiti Sains Malaysia, Penang, Malaysia

الملخص:

في هذا البحث أوجدنا الحلول لمعادلة من نوع $x^d=\beta$ في الزمرة المتناوبة A_n لكل من $\beta \in H_n \cap C^\alpha$ حيث $n \in \theta = \{1,2,5,6,10,14\}$, β عنصر في صف الترافق $A(\beta)$ في A_n و d هو عدد صحيح موجب، $H_n = \{C^\alpha \text{ of } S_n | n > 1,$ {مع كل الأجزاء α_k من α تكون مختلفة وفردي، C^α هو صف الترافق في S_n وشكل كل صف ترافق C^α يعتمد على التجزئة الدورية α لعناصرها. وفي اتجاه آخر، فإنه لأي تبديل λ في الزمرة المتناظرة S_n، إذا كان $\lambda \in C^\alpha$ و $\lambda \notin H_n \cap C^\alpha$ فإن C^α لا تنقسم إلى الصفين $C^{\alpha\pm}$ من A_n .
في هذا البحث تم تحديد عدد الحلول كما أن هذا العمل معزز بعدد من الأمثلة.

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Solving class equation $x^d = \beta$ in an alternating group for all $n \in \theta$ & $\beta \in H_n \cap C^\alpha$

Shuker Mahmood [a,b,*], **Andrew Rajah** [b]

[a] *Department of Mathematics, College of Science, University of Basra, Basra, Iraq*
[b] *School of Mathematical Sciences, Universiti Sains Malaysia, Penang, Malaysia*

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Abstract In this paper we find out the solutions to the class equation $x^d = \beta$ in the alternating group A_n for each $\beta \in H_n \cap C^\alpha$ and $n \in \theta = \{1, 2, 5, 6, 10, 14\}$, where β ranges over the conjugacy class $A(\beta)$ in A_n and d is a positive integer number, $H_n = \{C^\alpha \text{ of } S_n | n > 1$, with all parts α_k of α different and odd$\}$, C^α is conjugacy class of S_n and form each conjugacy class C^α depends on the cycle partition α of its elements. In another direction, for any permutation λ in the symmetric group S_n, if $\lambda \in C^\alpha$ and $\lambda \notin H_n \cap C^\alpha$, then C^α does not split into the two classes $C^{\alpha\pm}$ of A_n. Moreover, in the present research, the number of solutions is determined and this current work is supported by examples.

1. Introduction

If x is a solution of $x^n = \beta$, n is a positive integer and y is a conjugate to x, then y is a solution of $x^n = \lambda$, where λ is conjugate to β in an alternating group (or any finite group). We call $x^n = \beta$ a class equation in A_n, where β and x belong to conjugate classes in an alternating group. The Frobenius equation $x^d = c$ and conjugacy classes in finite groups were introduced by Frobenius (1903), and studied by many others, such as Lam (1988), Kimmerle and Sandling (1992), Mann and Martinesz (1996), Muller (2000),Takegahara (2002) and Eric (2007), who dealt with some types of finite groups, including finite cyclic groups, m-generated finite groups, and Wreath products of finite groups. Goldmann and Russell (2002) studied the computational complexity of solving systems of equations over a finite group G, where $x_1x_2x_3 \ldots x_h = 1_G$ is an equation over a finite group G. A study was introduced by Taban (2007) to solve the class equation $x^d = \beta$ in a symmetric group and explain the solutions using group-theoretic approach. This approach states that all pairs of permutations γ and β in a symmetric group are conjugates iff they have the same structure. However, this is not necessarily true in an alternating group A_n specially, at $n \in \theta$. Moreover, Montserrat and Ilva (2011) gave a description of the solution set of systems of equations over an equationally Noetherian free product of groups G by using an analogue of the Makanin–Razborov diagrams. Also, Gabor and Csaba (2012) show that the complexities of the equivalence and the equation solvability problems are not determined by the clone of the algebra, where they explain that by using alternating group on four elements. In any way in the current work, the conjugacy classes in an alternating group will be studied in detail when $n \in \theta$. Choose any $\beta \in S_n$ and write it as $\gamma_1, \gamma_2, \ldots, \gamma_{c(\beta)}$. With γ_i disjoint cycles of length α_i and $c(\beta)$ is the number of disjoint cycle factors including the 1-cycle of β.

* Corresponding author at: Department of Mathematics, College of Science, University of Basra, Basra, Iraq. Tel.: +964 7713144239.
E-mail addresses: shuker.alsalem@gmail.com (S. Mahmood), andy@cs.usm.my (A. Rajah).
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Since disjoint cycles commute, we can assume that $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_{c(\beta)}$. Therefore $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{c(\beta)})$ is a partition of n and it is called cycle type of β. Let $C^\alpha \subset S_n$ be the set of all elements with cycle type α, then we can determine the conjugate class of $\beta \in S_n$ by using cycle type of β, since each pair of λ and β in S_n are conjugate iff they have the same cycle type (Zeindler, 2010). Therefore, the number of conjugacy classes of S_n is the number of partitions of n. However, this is not necessarily true in an alternating group. Let $\beta = (124)$ and $\lambda = (142)$ be two permutations in S_4 that belong to the same conjugate $\alpha(\beta) = (\alpha_1(\beta), \alpha_2(\beta)) = (1,3) = (\alpha_1(\lambda), \alpha_2(\lambda)) = \alpha(\lambda)$, and they have the same cycle structure, but λ and β are not conjugate in A_4. Since $(k_1, k_2, \ldots, k_r)^{-1} = (k_r, \ldots, k_2, k_1)$ we obtain $\alpha(\beta) = \alpha(\beta^{-1})$, then every permutation in S_n is conjugate to its inverse. Thus we have S_n as an ambivalent group. This is not true for the alternating groups, where if we assume $\theta = \{1,2,5,6,10,14\}$ we have $(A_n, n \in \theta)$ as ambivalent groups and $(A_n, n \notin \theta)$ as not ambivalent groups. The main purpose of the present research is to solve and determine the number of solutions of the class equation $x^d = \beta$ (i.e find out the solution set $X = \{x \in A_n | x^d \in A(\beta)\}$ and the number of these solutions $|X|$) in the alternating group A_n for each $\beta \in H_n \cap C^\alpha$ and $n \in \theta$, where β ranges over the conjugacy class $A(\beta)$ in A_n and d is a positive integer number, $H_n = \{C^\alpha \text{ of } S_n | n > 1$, with all parts α_k of α different and odd$\}$. C^α is conjugacy class of S_n and form each conjugacy class C^α depends on the cycle partition α of its elements. If $\lambda \in C^\alpha$ and $\lambda \notin H_n \cap C^\alpha$, then C^α does not split into the two classes $C^{\alpha\pm}$ of A_n.

2. Definitions and notations

Definition 2.1. A partition α is a sequence of nonnegative integers $(\alpha_1, \alpha_2, \ldots)$ with $\alpha_1 \geqslant \alpha_2 \geqslant \cdots$, and $\sum_{i=1}^{\infty} \alpha_i < \infty$. The length $l(\alpha)$ and the size $|\alpha|$ of α are defined as $l(\alpha) = Max\{i \in N;\ \alpha_i \neq 0\}$ and $|\alpha| = \sum_{i=1}^{\infty} \alpha_i$ We set $\alpha \vdash n = \{\alpha \text{ partition}; |\alpha| = n\}$ for $n \in N$. An element of $\alpha \vdash n$ is called a partition of n (see Zeindler, 2010).

Remark 2.2. We only write the non zero components of a partition. Choose any $\beta \in S_n$ and write it as $\gamma_1\gamma_2 \ldots \gamma_{c(\beta)}$. With γ_i disjoint cycles of length α_i and $c(\beta)$ is the number of disjoint cycle factors including the 1-cycle of β. Since disjoint cycles commute, we can assume that $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_{c(\beta)}$. Therefore $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{c(\beta)})$ is a partition of n and each α_i is called part of α (see Zeindler, 2010).

Definition 2.3. We call the partition $\alpha = \alpha(\beta) = (\alpha_1(\beta), \alpha_2(\beta), \ldots, \alpha_{c(\beta)}(\beta))$ the cycle type of β (Zeindler, 2010).

Definition 2.4. Let α be a partition of n. We define $C^\alpha \subset S_n$ to be the set of all elements with cycle type α (Zeindler, 2010).

Definition 2.5. Let $\beta \in S_n$ be given. We define $c_m = c_m^{(n)} = c_m^{(n)}(\beta)$ to be the number of cycles of length m of β (Zeindler, 2010).

Remark 2.6.

(**1**) The relationship between partitions and c_m is as follows: if $\beta \in C^\alpha$ is given then $c_m^{(n)}(\beta) = |\{i : \alpha_i = m\}|$ (see Zeindler, 2010).

(**2**) The cardinality of each C^α can be found as follows: $|C^\alpha| = \frac{n!}{z_\alpha}$ with $z_\alpha = \prod_{r=1}^{n} r^{c_r}(c_r)!$ and $c_r = c_r^{(n)}(\beta) = |\{i : \alpha_i = r\}|$ (see Bump, 2004).

(**3**) $|C^\alpha(\beta)^+| = |C^\alpha(\beta)^-| = \frac{|C^\alpha(\beta)|}{2}$ (see James and Kerber, 1984). Hence, the number of the solutions for the class equation $x^d = \beta$ in A_n if exists is only $\frac{n!}{2z_{\alpha(\beta)}}$.

Definition 2.7. Let $\beta \in C^\alpha$, where β is a permutation in A_n. $A(\beta)$ conjugacy class of β in A_n is defined by

$$A(\beta) = \{\gamma \in A_n | \gamma = t\beta t^{-1};\ \text{for some } t \in A_n\}$$
$$= \begin{cases} C^\alpha, & (\text{if } \beta \notin H_n) \\ C^{\alpha+} \text{ or } C^{\alpha-}, & (\text{if } \beta \in H_n) \end{cases}$$

where $H_n = \{C^\alpha \text{ of } S_n | n > 1$, with all parts α_k of α different and odd$\}$ (Mahmood and Rajah, 2011).

Remark 2.8.

(1) $\beta \in H_n \Rightarrow \beta \in A_n$.
(2) $\beta \in C^\alpha \cap H_n^c \cap A_n \Rightarrow A(\beta) = C^\alpha$, where H_n^c is a complement of H_n.
(3) $\beta \in C^\alpha \cap H_n \Rightarrow C^\alpha$ splits into two classes $C^{\alpha\pm}$ of A_n.
(4) If $\beta, \lambda \in C^\alpha \cap H_n$, and $\lambda \in C^{\alpha+}$, then $A(\beta) = \begin{cases} C^{\alpha+} & \text{if } \beta \underset{A_n}{\approx} \lambda \\ C^{\alpha-} & \text{if } O.W \end{cases}$
(5) If $n \in \theta = \{1,2,5,6,10,14\}$, then for each $\beta \in A_n$ β is conjugate to β^{-1} in $A_n \left(\beta \underset{A_n}{\approx} \beta^{-1}\right)$ (Shuker et al., 2011).

Theorem 2.9. *If $\beta \in [A^a, B^b, \ldots, T^t]$ is a conjugacy class in symmetric group S_n, then $\beta^d \in [\Omega^{ra}, \phi^{sb}, \ldots, \psi^{tm}]$ where $gcd(d, A) = r$, $gcd(d, B) = s, \ldots, gcd(d, T) = m$ and $A = \Omega r$, $B = \phi s, \ldots, T = \psi m$ (Taban, 2007).*

Remark 2.10.

(1) If $x^d \in [B_1^{b_1}, B_2^{b_2}, \ldots, B_m^{b_m}]$, then we can find each solution for every part of $x^d \in [B_1^{b_1}], x^d \in [B_2^{b_2}], \ldots,$ and $x^d \in [B_m^{b_m}]$ separately, and then we collect all the solutions to find the solution of $x^d \in [B_1^{b_1}, B_2^{b_2}, \ldots, B_m^{b_m}]$ in S_n. Moreover, if there is no solution for at least one of the parts, then there is no solution of $x^d \in [B_1^{b_1}, B_2^{b_2}, \ldots, B_m^{b_m}]$ in S_n.
(2) Theorem 2.9 gives us all conjugacy classes of the form $[T_1^{t_1}, T_2^{t_2}, \ldots, T_m^{t_m}]$ which are belonging to the solution set of $x^d \in [B_1^{r_1t_1}, B_2^{r_2t_2}, \ldots, B_m^{r_mt_m}]$ in S_n, where $\gcd(d, T_1) = r_1, \gcd(d, T_2) = r_2, \ldots, \gcd(d, T_m) = r_m$ and $T_1 = B_1r_1$, $T_2 = B_2r_2, \ldots, T_m = B_mr_m$. But this does not give us all the solutions of $x^d \in [B_1^{r_1t_1}, B_2^{r_2t_2}, \ldots, B_m^{r_mt_m}]$ in S_n except when $t_1 = t_2 = \cdots = t_m = 1$ and $r_1 = r_2 = \cdots = r_m = 1$ (Taban, 2007).

3. Ambivalent alternating groups

The group in which each element is a conjugate of its inverse is called ambivalent group. Moreover, for all pairs of permutations γ and β in a symmetric group are conjugates iff they have the same structure. However, this is not necessarily true in an alternating group A_n specially, at $n \in \theta = \{1, 2, 5, 6, 10, 14\}$. In another direction, if $n \in \theta$. Then A_n is an ambivalent alternating group and form set H_n for each $n \in \theta$ can be summarized as follows:

1. $H_n = \phi$ and $H_n^c = S_n$ *if* $(n = 1, 2)$.
2. $H_5 = \{[5]\}$.
3. $H_6 = \{[1, 5]\}$.
4. $H_{10} = \{[1, 9], [3, 7]\}$.
5. $H_{14} = \{[1, 13], [5, 9], [3, 11]\}$.

Definition 3.1. Let $\gamma = (a_1, a_2, a_3, a_4, a_5) \in [5]$ of S_5 and $\beta = (b_1, b_2, b_3, b_4, b_5) \in [1, 5]$ of S_6. We define classes $[5]^{\pm}$ and $[1, 5]^{\pm}$ as following:

$A(\gamma) = [5]^{+} = \{\lambda \in [5] | \lambda = t\gamma t^{-1}$ for some $t \in A_5\}$,

$A\left(\overset{\#}{\gamma}\right) = [5]^{-} = \{\lambda \in [5] | \lambda = t\overset{\#}{\gamma} t^{-1}$ for some $t \in A_5\}$, where $\overset{\#}{\gamma} = (a_1, a_3, a_5, a_2, a_4)$,

$A(\beta) = [1,5]^{+} = \{\lambda \in [1,5] | \lambda = t\beta t^{-1}$ for some $t \in A_6\}$, and

$A\left(\overset{\#}{\beta}\right) = [1, 5]^{-} = \{\lambda \in [1, 5] | \lambda = t\overset{\#}{\beta} t^{-1}$ for some $t \in A_6\}$,

where $\overset{\#}{\beta} = (b_1, b_3, b_5, b_2, b_4)$.

Remark 3.2.

(i) Let $\gamma = (a_1, a_2, a_3, a_4, a_5) \in [5]$ of S_5 and $\beta = (b_1, b_2, b_3, b_4, b_5) \in [1, 5]$ of S_6, where $\overset{\#}{\gamma} = (a_1, a_3, a_5, a_2, a_4)$, $\overset{\#}{\beta} = (b_1, b_3, b_5, b_2, b_4)$, and d is a positive integer. Thus,

(1)	$\gamma^d = \gamma$ and $\beta^d = \beta \iff d \equiv 1 \pmod 5$
(2)	$\gamma^d = \overset{\#}{\gamma}$ and $\beta^d = \overset{\#}{\beta} \iff d \equiv 2 \pmod 5$
(3)	$\gamma^d = \overset{\#}{\gamma}^{-1}$ and $\beta^d = \overset{\#}{\beta}^{-1} \iff d \equiv 3 \pmod 5$
(4)	$\gamma^d = \gamma^{-1}$ and $\beta^d = \beta^{-1} \iff d \equiv 4 \pmod 5$

(ii) $A(\gamma) = A(\gamma^{-1})$, $A\left(\overset{\#}{\gamma}\right) = A\left(\overset{\#}{\gamma}^{-1}\right)$ in A_5, and $A(\beta) = A(\beta^{-1})$, $A\left(\overset{\#}{\beta}\right) = A\left(\overset{\#}{\beta}^{-1}\right)$ in A_6, [given that A_n, for $(n = 5,6)$ are ambivalent groups]. $\lambda \in A\left(\overset{\#}{\gamma}\right) \cap [5] \lambda \in A\left(\overset{\#}{\beta}\right) \cap [1, 5]$

(iii) $A(\lambda) = A(\lambda^{-1}) = [5]^{+}$ if $\lambda \in A(\gamma) \cap [5]$, and $A(\lambda) = A(\lambda^{-1}) = [1, 5]^{+}$ if $\lambda \in A(\beta) \cap [1, 5]$.

(iv) $A\left(\overset{\#}{\lambda}\right) = A\left(\overset{\#}{\lambda}^{-1}\right) = [5]^{-}$ if $\lambda \in A\left(\overset{\#}{\gamma}\right) \cap [5]$, and $A\left(\overset{\#}{\lambda}\right) = A\left(\overset{\#}{\lambda}^{-1}\right) = [1, 5]^{-}$ if $\lambda \in A\left(\overset{\#}{\beta}\right) \cap [1, 5]$.

Lemma 3.3. *Let $L = \{m \in N | m \equiv q \pmod 5$ for some $q = 1,4\}$. If d is a positive integer such that $gcd(d,5) = 1$ and $\beta \in [5]$ of S_5, then the solutions of $x^d \in A(\beta)$ in A_5 are*

1. $A(\beta)$, if $d \in L$.
2. $A\left(\overset{\#}{\beta}\right)$, if $d \notin L$.

Proof. Given that $\beta \in [5] \cap H_5$, [5] splits into two classes $A(\beta)$ and $A\left(\overset{\#}{\beta}\right)$ of A_5. Moreover, $\gcd(d, 5) = 1$. Then, by (2.10), $[5] = A(\beta) \cup A\left(\overset{\#}{\beta}\right)$ is a solution set of $x^d \in [5] = A(\beta) \cup A\left(\overset{\#}{\beta}\right)$ in S_5. However, $A(\beta) \cap A\left(\overset{\#}{\beta}\right) = \phi$. Hence, for each $\pi \in [5] \Rightarrow \left(\pi \in A(\beta) \& \pi \notin A\left(\overset{\#}{\beta}\right)\right)$ or $\left(\pi \in A\left(\overset{\#}{\beta}\right) \& \pi \notin A(\beta)\right)$.

(1) Assume $d \in L$. If $\pi \in A(\beta)$, then $(\pi \underset{A_5}{\approx} \beta)\pi$ is conjugate to β in A_5. However, $\pi^d \underset{A_5}{\approx} \pi$ (because $d \in L) \Rightarrow \pi^d \underset{A_5}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A\left(\overset{\#}{\beta}\right)$. If $\pi \in A\left(\overset{\#}{\beta}\right)$, then $\left(\pi \underset{A_5}{\approx} \overset{\#}{\beta}\right)$. However, $\pi^d \underset{A_5}{\approx} \pi$ (because $d \in L) \Rightarrow \pi^d \underset{A_5}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A\left(\overset{\#}{\beta}\right) \& \pi^d \notin A(\beta)$. Then, the solution set of $x^d \in A(\beta)$ in A_5 is $A(\beta)$.

(2) Assume $d \notin L$. If $\pi \in A(\beta)$, then $\left(\pi \underset{A_5}{\approx} \beta\right) \Rightarrow \overset{\#}{\pi} \underset{A_5}{\approx} \overset{\#}{\beta}$. However, $\pi^d \underset{A_5}{\approx} \overset{\#}{\pi}$ (because $d \notin L) \Rightarrow \pi^d \underset{A_5}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A\left(\overset{\#}{\beta}\right) \& \pi^d \notin A(\beta)$. If $\pi \in A\left(\overset{\#}{\beta}\right) \Rightarrow \left(\pi \underset{A_5}{\approx} \overset{\#}{\beta}\right) \Rightarrow \overset{\#}{\pi} \underset{A_5}{\approx} \beta$. However, $\pi^d \underset{A_5}{\approx} \overset{\#}{\pi}$ (because $d \notin L) \Rightarrow \pi^d \underset{A_5}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A\left(\overset{\#}{\beta}\right)$. Then, the solution set of $x^d \in A(\beta)$ in A_5 is $A\left(\overset{\#}{\beta}\right)$. □

Example 3.4. Find the solutions of $x^{23} \in A(5\ 3\ 2\ 4\ 1)$ in A_5 and the number of the solutions.

<u>**Solution:**</u> Since $\beta = (5\ 3\ 2\ 4\ 1) \in H_5$, $\gcd(23, 5) = 1$ and $23 \notin L$, then the solution set is $A(5\ 2\ 1\ 3\ 4)$ and the number of the solutions is $\frac{|[5]|}{2} = \frac{5!}{2 \times 5} = 12$, where $A(5\ 2\ 1\ 3\ 4) = \{(1\ 2\ 3\ 5\ 4), (1\ 5\ 4\ 2\ 3), (1\ 2\ 5\ 4\ 3), (1\ 3\ 2\ 4\ 5), (1\ 4\ 2\ 5\ 3), (1\ 3\ 5\ 2\ 4), (1\ 2\ 4\ 3\ 5), (1\ 4\ 5\ 3\ 2), (1\ 4\ 3\ 2\ 5), (1\ 5\ 2\ 3\ 4), (1\ 5\ 3\ 4\ 2), (1\ 3\ 4\ 5\ 2)\}$.

Theorem 3.5. *Let $L = \{m \in N | m \equiv q \pmod 5$ for some $q = 1,4\}$. If d is a positive integer such that $gcd(d,5) = 1$ and $\beta \in [1,5]$ of S_6, then the solutions of $x^d \in A(\beta)$ in A_6 are*

1. $A(\beta)$, if $d \in L$.
2. $A\left(\overset{\#}{\beta}\right)$, if $d \notin L$.

Proof. Given that $\beta \in [1, 5] \cap H_6$,[1, 5] splits into two classes $A(\beta)$and $A\left(\overset{\#}{\beta}\right)$ of A_6. Moreover, $\gcd(d, 5) = 1$. Then, by

(2.10), $[1,5] = A(\beta) \cup A\left(\overset{\#}{\beta}\right)$ is a solution set of $x^d \in [1,5] = A(\beta) \cup A\left(\overset{\#}{\beta}\right)$ in S_6. However, $A(\beta) \cap A\left(\overset{\#}{\beta}\right) = \phi$. Hence, for each $\pi \in [1,5] \Rightarrow \left(\pi \in A(\beta) \& \pi \notin A\left(\overset{\#}{\beta}\right)\right)$ or $\left(\pi \in A\left(\overset{\#}{\beta}\right) \& \pi \notin A(\beta)\right)$.

(1) Assume $d \in L$. If $\pi \in A(\beta)$, then $(\pi \underset{A_6}{\approx} \beta)\pi$ is conjugate to β in A_6. However, $\pi^d \underset{A_6}{\approx} \pi$ (because $d \in L) \Rightarrow \pi^d \underset{A_6}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A\left(\overset{\#}{\beta}\right)$. If $\pi \in A\left(\overset{\#}{\beta}\right)$, then $\left(\pi \underset{A_6}{\approx} \overset{\#}{\beta}\right)$. However, $\pi^d \underset{A_6}{\approx} \pi$ (because $d \in L) \Rightarrow \pi^d \underset{A_6}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A\left(\overset{\#}{\beta}\right) \& \pi^d \notin A(\beta)$. Then, the solution set of $x^d \in A(\beta)$ in A_6 is $A(\beta)$.

(2) Assume $d \notin L$. If $\pi \in A(\beta)$, then $\left(\pi \underset{A_6}{\approx} \beta\right) \Rightarrow \overset{\#}{\pi} \underset{A_6}{\approx} \overset{\#}{\beta}$. However, $\pi^d \underset{A_6}{\approx} \overset{\#}{\pi}$ (because $d \notin L) \Rightarrow \pi^d \underset{A_6}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A\left(\overset{\#}{\beta}\right) \& \pi^d \notin A(\beta)$. If $\pi \in A\left(\overset{\#}{\beta}\right) \Rightarrow (\pi \underset{A_6}{\approx} \overset{\#}{\beta}) \Rightarrow \overset{\#}{\pi} \underset{A_6}{\approx} \beta$. However, $\pi^d \underset{A_6}{\approx} \overset{\#}{\pi}$ (because $d \notin L) \Rightarrow \pi^d \underset{A_6}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A\left(\overset{\#}{\beta}\right)$. Then, the solution set of $x^d \in A(\beta)$ in A_6 is $A\left(\overset{\#}{\beta}\right)$. □

Example 3.6. Find the solutions of $x^{19} \in A(2\ 4\ 5\ 6\ 3)$ in A_6 and the number of solutions.

Solution: In as much as $\beta = (2\ 4\ 5\ 6\ 3) \in H_6 = \{[1,5]\}$, $\gcd(19,5) = 1$, and $19 \in L$, then the solution set is $A(2\ 4\ 5\ 6\ 3)$, and the number of the solutions is $\frac{\|[1,5]\|}{2} = \frac{6!}{2\times 5} = 72$, where

$A(2\ 4\ 5\ 6\ 3)$	= {(1 5 4 3 2), (1 2 3 4 5), (1 2 5 3 4), (1 3 2 5 4), (1 4 2 3 5), (1 3 5 4 2),
	(1 4 5 2 3), (1 5 2 4 3), (1 5 3 2 4), (1 3 4 2 5), (1 2 4 3 6), (1 2 5 6 3), (1 2 5 4 6),
	(1 4 3 6 5), (1 4 2 6 3), (14 5 6 2), (1 5 4 6 3), (1 5 4 2 6), (1 5 6 3 4), (1 5 6 2 3),
	(1 5 2 6 4), (1 5 2 3 6), (1 5 6 4 2), (1 6 5 2 4), (1 6 3 5 4), (1 6 3 4 2), (2 3 4 6 5),
	(1 5 3 6 2), (1 6 4 5 2), (1 6 5 3 2), (1 6 4 2 3), (1 6 4 3 5), (1 6 5 4 3), (1 6 2 4 5),
	(1 3 6 4 5), (1 3 2 6 5), (1 4 5 3 6), (1 4 6 3 2), (1 4 6 2 5), (1 4 3 2 6), (2 3 6 5 4),
	(2 4 6 3 5), (2 6 3 4 5), (2 6 4 5 3), (1 5 3 4 6), (1 4 3 5 2), (1 2 4 5 3), (1 2 4 6 5)
	(1 2 3 5 6), (1 2 3 6 4), (1 2 6 4 3), (1 2 6 5 4), (1 2 6 3 5), (1 3 6 5 2), (1 3 4 6 2),
	(1 3 2 4 6), (1 3 4 5 6), (1 3 5 6 4), (1 3 5 2 6), (1 3 6 2 4), (1 4 2 5 6), (1 4 6 5 3),
	(1 6 2 3 4), (1 6 2 5 3), (1 6 3 2 5), (2 4 5 6 3), (2 5 6 4 3), (2 6 5 3 4), (2 4 3 5 6),
	(2 5 4 3 6), (2 3 5 4 6), (2 5 3 6 4)}

Definition 3.7. Let $\beta = \gamma\lambda \in [3,7]$ of S_{10}, where $\gamma = (b_1, b_2, b_3)$, $\lambda = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$. We define classes $[3,7]^{\pm}$ of A_{10} by:

$$A(\beta) = [3,7]^{+} = \{\mu \in [3,7] \mid \mu = t\beta t^{-1} \text{ for some } t \in A_{10}\}$$

and $A\left(\overset{\#}{\beta}\right) = [3,7]^{-} = \left\{\mu \in [3,7] \mid \mu = t\overset{\#}{\beta}t^{-1} \text{ for some } t \in A_{10}\right\}$, where $\overset{\#}{\beta} = \gamma\bar{\lambda}$ and $\bar{\lambda} = (a_1, a_4, a_7, a_3, a_6, a_2, a_5)$.

Remark 3.8

(i) Let $\beta = \gamma\lambda \in [3,7]$ of S_{10}, where $\gamma = (b_1, b_2, b_3)$, $\lambda = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$, $\bar{\lambda} = (a_1, a_4, a_7, a_3, a_6, a_2, a_5)$, $\bar{\bar{\lambda}} = (a_1, a_3, a_5, a_7, a_2, a_4, a_6)$, and d is a positive integer number. Thus,

(1) $\beta^d = \beta \iff d \equiv 1 \pmod{21}$
(2) $\beta^d = \gamma^{-1}\bar{\bar{\lambda}} \iff d \equiv 2 \pmod{21}$
(3) $\beta^d = \gamma\lambda^{-1} \iff d \equiv 4 \pmod{21}$
(4) $\beta^d = \gamma^{-1}\bar{\bar{\lambda}}^{-1} \iff d \equiv 5 \pmod{21}$
(5) $\beta^d = \gamma^{-1}\lambda \iff d \equiv 8 \pmod{21}$
(6) $\beta^d = \gamma\bar{\lambda} \iff d \equiv 10 \pmod{21}$
(7) $\beta^d = \gamma^{-1}\bar{\lambda}^{-1} \iff d \equiv 11 \pmod{21}$
(8) $\beta^d = \gamma\lambda^{-1} \iff d \equiv 13 \pmod{21}$
(9) $\beta^d = \gamma\bar{\bar{\lambda}} \iff d \equiv 16 \pmod{21}$
(10) $\beta^d = \gamma^{-1}\bar{\lambda} \iff d \equiv 17 \pmod{21}$
(11) $\beta^d = \gamma\bar{\bar{\lambda}}^{-1} \iff d \equiv 19 \pmod{21}$
(12) $\beta^d = \beta^{-1} \iff d \equiv 20 \pmod{21}$

(ii) $A(\beta) = A(\beta^{-1})$, $A(\gamma^{-1}\bar{\bar{\lambda}}) = A(\gamma\bar{\bar{\lambda}}^{-1})$, $A(\gamma\bar{\lambda}^{-1}) = A(\gamma^{-1}\bar{\lambda})$, $A(\gamma^{-1}\bar{\bar{\lambda}}^{-1}) = A(\gamma\bar{\bar{\lambda}})$, $A(\gamma^{-1}\lambda) = A(\gamma\lambda^{-1})$, $A(\gamma\bar{\lambda}) = A(\gamma^{-1}\bar{\lambda}^{-1})$ [given that ambivalent group]

(iii) $A(\beta) = A(\gamma^{-1}\bar{\lambda})$ [because, $\exists t = (b_2, b_3)(a_1, a_4, a_5, a_3, a_7, a_6) \in A_{10}$ such that $t\beta t^{-1} = \gamma^{-1}\bar{\lambda}$].

(iv) $A(\beta) = A(\gamma\bar{\bar{\lambda}})$ [because, $\exists t = (a_1, a_3, a_4)(a_7, a_6, a_2) \in A_{10}$ such that $t\beta t^{-1} = \gamma\bar{\bar{\lambda}}$].

(v) $A(\gamma\bar{\bar{\lambda}}^{-1}) = A(\gamma\bar{\lambda})$ [because, $\exists t = (a_1, a_4, a_2)(a_3, a_5, a_6) \in A_{10}$ such that $t\gamma\bar{\bar{\lambda}}^{-1}t^{-1} = \gamma\bar{\lambda}$].

(vi) $A(\gamma\bar{\lambda}^{-1}) = A(\gamma\lambda^{-1})$ [because, $\exists t = (a_1, a_6, a_5)(a_2, a_3, a_7) \in A_{10}$ such that $t\gamma\lambda^{-1}t^{-1} = \gamma\bar{\bar{\lambda}}^{-1}$].

Lemma 3.9. *Let $L = = \{m \in N \mid m \equiv q \pmod{21}$ for some $q = 1, 4, 5, 16, 17, 20\}$. If d is a positive integer such that $\gcd(d,3) = 1$ & $\gcd(d,7) = 1$ and $\beta \in [3,7]$ of S_{10}, then the solutions of $x^d \in A(\beta)$ in A_{10} are*

1. $A(\beta)$, if $d \in L$.
2. $A\left(\overset{\#}{\beta}\right)$, if $d \notin L$.

Proof. Given that $\beta \in [3,7] \cap H_{10}$, $[3,7]$ splits into two classes $A(\beta)$ and $A\left(\overset{\#}{\beta}\right)$ of A_{10}. Moreover, $\gcd(d,3) = 1$ and $\gcd(d,7) = 1$. Then, by (2.10), $[3,7] = A(\beta) \cup A\left(\overset{\#}{\beta}\right)$ is a solution set of $x^d \in [3,7] = A(\beta) \cup A\left(\overset{\#}{\beta}\right)$ in S_{10}. However, $A(\beta) \cap A\left(\overset{\#}{\beta}\right) = \phi$. Hence, for each $\pi \in [3,7] \Rightarrow \left(\pi \in A(\beta) \& \pi \notin A\left(\overset{\#}{\beta}\right)\right)$ or $\left(\pi \in A\left(\overset{\#}{\beta}\right) \& \pi \notin A(\beta)\right)$.

(1) Assume $d \in L$. If $\pi \in A(\beta)$, then $(\pi \underset{A_{10}}{\approx} \beta)\pi$ is conjugate to β in A_{10}. However, $\pi^d \underset{A_{10}}{\approx} \pi$ (because $d \in L) \Rightarrow \pi^d \underset{A_{10}}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A\left(\overset{\#}{\beta}\right)$. If

$\pi \in A\left(\overset{\#}{\beta}\right)$, then $(\pi \underset{A_{10}}{\approx} \overset{\#}{\beta})$. However, $\pi^d \underset{A_{10}}{\approx} \pi$ (because $d \in L$) $\Rightarrow \pi^d \underset{A_{10}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A\left(\overset{\#}{\beta}\right) \& \pi^d \notin A(\beta)$. Then, the solution set of $x^d \in A(\beta)$ in A_{10} is $A(\beta)$.

(2) Assume $d \notin L$. If $\pi \in A(\beta)$, then $(\pi \underset{A_{10}}{\approx} \beta) \Rightarrow \overset{\#}{\pi} \underset{A_{10}}{\approx} \overset{\#}{\beta}$. However, $\pi^d \underset{A_{10}}{\approx} \overset{\#}{\pi}$ (because $d \notin L$) $\Rightarrow \pi^d \underset{A_{10}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A\left(\overset{\#}{\beta}\right)$ $\& \pi^d \notin A(\beta)$. If $\pi \in A\left(\overset{\#}{\beta}\right) \Rightarrow (\pi \underset{A_{10}}{\approx} \overset{\#}{\beta}) \Rightarrow \overset{\#}{\pi} \underset{A_{10}}{\approx} \beta$. However, $\pi^d \underset{A_{10}}{\approx} \overset{\#}{\pi}$ (because $d \notin L$) $\Rightarrow \pi^d \underset{A_{10}}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A\left(\overset{\#}{\beta}\right)$. Then, the solution set of $x^d \in A(\beta)$ in A_{10} is $A\left(\overset{\#}{\beta}\right)$. □

Definition 3.10. Let $\beta = [1,9]$ of S_{10}, where $\beta = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$. We define class $[1,9]^+$ of A_{10} by $A(\beta) = [1,9]^+ = \{\mu \in [1,9] \mid \mu = t\beta t^{-1} \text{ for some } t \in A_{10}\}$

Remark 3.11.

(i) $[1,9]^- = [1,9] - A(\beta) = \{\mu \in [1,9] \mid \mu \neq t\beta t^{-1} \text{ for all } t \in A_{10}\}$

(ii) Let $\beta = [1,9]$ of S_{10} where $\beta = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$, $\bar{\beta} = (a_1, a_3, a_5, a_7, a_9, a_2, a_4, a_6, a_8)$, $\bar{\bar{\beta}} = (a_1, a_5, a_9, a_4, a_8, a_3, a_7, a_2, a_6)$, and d is a positive integer number. Thus,

(1)	$\beta^d = \beta \Longleftrightarrow d \equiv 1 \pmod 9$
(2)	$\beta^d = \bar{\beta} \Longleftrightarrow d \equiv 2 \pmod 9$
(3)	$\beta^d = \bar{\bar{\beta}} \Longleftrightarrow d \equiv 4 \pmod 9$
(4)	$\beta^d = \bar{\bar{\beta}}^{-1} \Longleftrightarrow d \equiv 5 \pmod 9$
(5)	$\beta^d = \bar{\beta}^{-1} \Longleftrightarrow d \equiv 7 \pmod 9$
(6)	$\beta^d = \beta^{-1} \Longleftrightarrow d \equiv 8 \pmod 9$

(iii) $A(\beta) = A(\beta^{-1})$, $A(\bar{\beta}) = A(\bar{\beta}^{-1})$, $A(\bar{\bar{\beta}}) = A(\bar{\bar{\beta}}^{-1})$ [because A_{10} is an ambivalent group].

(iv) $A(\beta) = A(\bar{\beta})$ [because, $\exists t = (b_3, b_6)(a_1, a_5, a_7, a_8, a_4, a_2) \in A_{10}$ such that $t\beta t^{-1} = \bar{\beta}$].

(v) $A(\beta^{-1}) = A(\bar{\bar{\beta}})$ [because, $\exists t = (b_3, b_9)(a_1, a_8, a_7, a_2, a_4, a_5) \in A_{10}$ such that $t\bar{\bar{\beta}}t^{-1} = \beta^{-1}$].

Theorem 3.12. *Let $\beta = [1,9]$ of S_{10}. If d is a positive integer such that $\gcd(d,9) = 1$, then the solution of $x^d \in A(\beta)$ in A_{10} is $A(\beta)$.*

Proof. Given that $\beta \in [1,9] \cap H_{10}$, $[1,9]$ splits into two classes $A(\beta)$ and $[1,9]^-$ of A_{10}. Moreover, $\gcd(d,9) = 1$. So by (2.10), the solution set of $x^d \in [1,9]$ in S_{10} is $[1,9]$. For each $\lambda \in [1,9]$, $(\lambda \in A(\beta))$ or $(\lambda \notin A(\beta))$. If $\lambda \in A(\beta)$, then $\lambda \underset{A_{10}}{\approx} \beta \Rightarrow \lambda^d \underset{A_{10}}{\approx} \beta^d$. However, $\beta^d \underset{A_{10}}{\approx} \beta \Rightarrow \lambda^d \underset{A_{10}}{\approx} \beta \Rightarrow \lambda^d \in A(\beta)$.

If $\lambda \notin A(\beta)$, assume $\lambda^d \in A(\beta) \Rightarrow \lambda^d \underset{A_{10}}{\approx} \beta$. But $\beta \underset{A_{10}}{\approx} \beta^d \Rightarrow \lambda^d \underset{A_{10}}{\approx} \beta^d \Rightarrow \lambda \underset{A_{10}}{\approx} \beta \Rightarrow \lambda \in A(\beta)$ which is contradiction. Then the solution of $x^d \in A(\beta)$ in A_{10} is $A(\beta)$. □

Definition 3.13. Let $\beta = \gamma\lambda \in [5,9]$ of S_{14}, where $\lambda = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ and $\gamma = (b_1, b_2, b_3, b_4, b_5)$, we define classes $[5,9]\pm$ of A_{14} by:

$$A(\beta) = [5,9]^+ = \{\mu \in [5,9] \mid \mu = t\beta t^{-1} \text{ for some } t \in A_{14}\}$$

and

$$A\left(\overset{\#}{\beta}\right) = [5,9]^- = \{\mu \in [5,9] \mid \mu = t\overset{\#}{\beta}t^{-1} \text{ for some } t \in A_{14}\},$$

where $\overset{\#}{\beta} = \bar{\gamma}\lambda$ and $\bar{\gamma} = (b_1, b_3, b_5, b_2, b_4)$.

Remark 3.14.

(i) Let $\beta = \gamma\lambda \in [5,9]$ of S_{14}, where $\gamma = (b_1, b_2, b_3, b_4, b_5)$, $\bar{\gamma} = (b_1, b_3, b_5, b_2, b_4)$ $\lambda = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$, $\bar{\lambda} = (a_1, a_3, a_5, a_7, a_9, a_2, a_4, a_6, a_8)$, $\bar{\bar{\lambda}} = (a_1, a_5, a_9, a_4, a_8, a_3, a_7, a_2, a_6)$ and d is a positive integer number. Thus,

(1)	$\beta^d = \beta \Longleftrightarrow d \equiv 1 \pmod{45}$
(2)	$\beta^d = \bar{\gamma}\bar{\lambda} \Longleftrightarrow d \equiv 2 \pmod{45}$
(3)	$\beta^d = \gamma^{-1}\bar{\bar{\lambda}} \Longleftrightarrow d \equiv 4 \pmod{45}$
(4)	$\beta^d = \bar{\gamma}\bar{\lambda}^{-1} \Longleftrightarrow d \equiv 7 \pmod{45}$
(5)	$\beta^d = \bar{\gamma}^{-1}\lambda^{-1} \Longleftrightarrow d \equiv 8 \pmod{45}$
(6)	$\beta^d = \gamma\bar{\lambda} \Longleftrightarrow d \equiv 11 \pmod{45}$
(7)	$\beta^d = \bar{\gamma}^{-1}\bar{\bar{\lambda}} \Longleftrightarrow d \equiv 13 \pmod{45}$
(8)	$\beta^d = \gamma^{-1}\bar{\bar{\lambda}}^{-1} \Longleftrightarrow d \equiv 14 \pmod{45}$
(9)	$\beta^d = \gamma\bar{\lambda}^{-1} \Longleftrightarrow d \equiv 16 \pmod{45}$
(10)	$\beta^d = \bar{\gamma}\lambda^{-1} \Longleftrightarrow d \equiv 17 \pmod{45}$
(11)	$\beta^d = \gamma^{-1}\lambda \Longleftrightarrow d \equiv 19 \pmod{45}$
(12)	$\beta^d = \bar{\gamma}\bar{\bar{\lambda}} \Longleftrightarrow d \equiv 22 \pmod{45}$
(13)	$\beta^d = \bar{\gamma}^{-1}\bar{\bar{\lambda}}^{-1} \Longleftrightarrow d \equiv 23 \pmod{45}$
(14)	$\beta^d = \gamma\lambda^{-1} \Longleftrightarrow d \equiv 26 \pmod{45}$
(15)	$\beta^d = \bar{\gamma}^{-1}\lambda^{-1} \Longleftrightarrow d \equiv 28 \pmod{45}$
(16)	$\beta^d = \gamma^{-1}\bar{\lambda} \Longleftrightarrow d \equiv 29 \pmod{45}$
(17)	$\beta^d = \gamma\bar{\bar{\lambda}} \Longleftrightarrow d \equiv 31 \pmod{45}$
(18)	$\beta^d = \bar{\gamma}\bar{\bar{\lambda}}^{-1} \Longleftrightarrow d \equiv 32 \pmod{45}$
(19)	$\beta^d = \gamma^{-1}\bar{\lambda}^{-1} \Longleftrightarrow d \equiv 34 \pmod{45}$
(20)	$\beta^d = \bar{\gamma}\lambda \Longleftrightarrow d \equiv 37 \pmod{45}$
(21)	$\beta^d = \bar{\gamma}^{-1}\bar{\lambda} \Longleftrightarrow d \equiv 38 \pmod{45}$
(22)	$\beta^d = \gamma\bar{\bar{\lambda}}^{-1} \Longleftrightarrow d \equiv 41 \pmod{45}$
(23)	$\beta^d = \bar{\gamma}^{-1}\bar{\lambda}^{-1} \Longleftrightarrow d \equiv 43 \pmod{45}$
(24)	$\beta^d = \gamma^{-1}\lambda^{-1} \Longleftrightarrow d \equiv 44 \pmod{45}$

(ii) $A(\beta) = A(\beta^{-1})$, $A(\bar{\gamma}\bar{\lambda}) = A(\bar{\gamma}^{-1}\bar{\lambda}^{-1})$, $A(\bar{\gamma}\bar{\bar{\lambda}}) = A(\bar{\gamma}^{-1}\bar{\bar{\lambda}}^{-1})$, $A(\bar{\gamma}\lambda) = A(\bar{\gamma}^{-1}\lambda^{-1})$, $A(\gamma\bar{\lambda}) = A(\gamma^{-1}\bar{\lambda}^{-1})$, $A(\gamma\bar{\bar{\lambda}}) = A(\gamma^{-1}\bar{\bar{\lambda}}^{-1})$, $A(\bar{\gamma}^{-1}\lambda) = A(\bar{\gamma}\lambda^{-1})$, $A(\bar{\gamma}^{-1}\bar{\lambda}) = A(\bar{\gamma}\bar{\lambda}^{-1})$, $A(\bar{\gamma}^{-1}\bar{\bar{\lambda}}) = A(\bar{\gamma}\bar{\bar{\lambda}}^{-1})$, $A(\gamma^{-1}\lambda) = A(\gamma\lambda^{-1})$, $A(\gamma^{-1}\bar{\lambda}) = A(\gamma\bar{\lambda}^{-1})$, $A(\gamma^{-1}\bar{\bar{\lambda}}) = A(\gamma\bar{\bar{\lambda}}^{-1})$ [given that A_{14} is an ambivalent group].

(iii) $A(\beta) = A(\gamma\bar{\lambda})$ [because, $\exists\ t = (a_3, a_6)(a_1, a_5, a_7, a_8, a_4, a_2) \in A_{14}$ such that $t\beta t^{-1} = \gamma\bar{\lambda}$]. Also, $A(\bar{\gamma}\bar{\lambda}) = A(\bar{\gamma}\lambda)$, $A(\bar{\gamma}^{-1}\lambda) = A(\bar{\gamma}^{-1}\bar{\lambda})$, $A(\gamma^{-1}\lambda) = A(\gamma^{-1}\bar{\lambda})$.

(iv) $A(\beta^{-1}) = A(\gamma^{-1}\bar{\bar{\lambda}})$ [because, $\exists t = (a_3, a_9)(a_1, a_8, a_7, a_2, a_4, a_5) \in A_{14}$ such that $t\gamma^{-1}\bar{\bar{\lambda}}t^{-1} = \beta^{-1}$]. Also, $A(\gamma\bar{\bar{\lambda}}) = A(\gamma\lambda^{-1})$, $A(\bar{\gamma}\bar{\bar{\lambda}}) = A(\bar{\gamma}\lambda^{-1})$, $A(\bar{\gamma}^{-1}\bar{\bar{\lambda}}) = A(\bar{\gamma}^{-1}\lambda^{-1})$

(v) $A(\beta^{-1}) = A(\gamma\lambda^{-1})$ [because, $\exists$ $t = (b_2, b_5)(b_3, b_4) \in A_{14}$ such that $t\gamma\lambda^{-1}t^{-1} = \beta^{-1}$]. Also, $A(\gamma\bar{\lambda}) = A(\gamma^{-1}\bar{\lambda})$, $A(\gamma\bar{\bar{\lambda}}) = A(\gamma^{-1}\bar{\bar{\lambda}})$.

Theorem 3.15. *Let $L = \{m \in N | m \equiv q \ (mod\ 45)$ for some $q = 1,4,11,14,16,19, 26,29,31,34,41,44\}$. If d is a positive integer such that $gcd(d,5) = 1$ & $gcd(d,9) = 1$ and $\beta \in [5,9]$ of S_{14}, then the solutions of $x^d \in A(\beta)$ in A_{14} are*

1. $A(\beta)$, if $d \in L$.
2. $A\left(\overset{\#}{\beta}\right)$, if $d \notin L$.

Proof. Given that $\beta \in [5,9] \cap H_{14}$, $[5,9]$ splits into two classes $A(\beta)$ and $A\left(\overset{\#}{\beta}\right)$ of A_{14}. Moreover, $\gcd(d,5) = 1$ and $\gcd(d,9) = 1$. So by (2.10), $[5,9] = A(\beta) \cup A\left(\overset{\#}{\beta}\right)$ is a solution set of $x^d \in [5,9] = A(\beta) \cup A\left(\overset{\#}{\beta}\right)$ in S_{14}. However, $A(\beta) \cap A\left(\overset{\#}{\beta}\right) = \phi$. Hence, for each $\pi \in [5,9] \Rightarrow \left(\pi \in A(\beta) \& \pi \notin A\left(\overset{\#}{\beta}\right)\right)$ or $\left(\pi \in A\left(\overset{\#}{\beta}\right) \& \pi \notin A(\beta)\right)$.

(1) Assume $d \in L$. If $\pi \in A(\beta)$, then $\left(\pi \underset{A_{14}}{\approx} \beta\right)\pi$ is conjugate to β in A_{14}. However, $\pi^d \underset{A_{14}}{\approx} \pi$ (because $d \in L$) $\Rightarrow \pi^d \underset{A_{14}}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A\left(\overset{\#}{\beta}\right)$. If $\pi \in A\left(\overset{\#}{\beta}\right)$, then $\left(\pi \underset{A_{14}}{\approx} \overset{\#}{\beta}\right)$. However, $\pi^d \underset{A_{14}}{\approx} \pi$ (because $d \in L) \Rightarrow \pi^d \underset{A_{14}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A\left(\overset{\#}{\beta}\right) \& \pi^d \notin A(\beta)$. Then, the solution set of $x^d \in A(\beta)$ in A_{14} is $A(\beta)$.

(2) Assume $d \notin L$. If $\pi \in A(\beta)$, then $\left(\pi \underset{A_{14}}{\approx} \beta\right) \Rightarrow \overset{\#}{\pi} \underset{A_{14}}{\approx} \overset{\#}{\beta}$. However, $\pi^d \underset{A_{14}}{\approx} \overset{\#}{\pi}$ (because $d \notin L) \Rightarrow \pi^d \underset{A_{14}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A\left(\overset{\#}{\beta}\right) \& \pi^d \notin A(\beta)$. If $\pi \in A\left(\overset{\#}{\beta}\right) \Rightarrow \left(\pi \underset{A_{14}}{\approx} \overset{\#}{\beta}\right) \Rightarrow \overset{\#}{\pi} \underset{A_{14}}{\approx} \beta$. However, $\pi^d \underset{A_{14}}{\approx} \overset{\#}{\pi}$ (because $d \notin L$) $\Rightarrow \pi^d \underset{A_{14}}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A\left(\overset{\#}{\beta}\right)$. Then, the solution set of $x^d \in A(\beta)$ in A_{14} is $A\left(\overset{\#}{\beta}\right)$. $\square$

Definition 3.16. Let $\beta \in [1,13]$ of S_{14}, where $\beta = (a_1, a_2, a_3, a_4,$- $= (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13})$. We define classes $[1,13]^{\pm}$ of A_{14} by:

$A(\beta) = [1,13]^{+} = \{\mu \in [1,13] | \mu = t\beta t^{-1}$ for some $t \in A_{14}\}$ and $A\left(\overset{\#}{\beta}\right) = [1,13]^{-} = \{\mu \in [1,13] | \mu = t\overset{\#}{\beta}t^{-1}$ for some $t \in A_{14}\}$,

where $\overset{\#}{\beta} = (a_1, a_3, a_5, a_7, a_9, a_{11}, a_{13}, a_2, a_4, a_6, a_8, a_{10}, a_{12})$.

Remark 3.17.

(i) Let $\beta \in [1,13]$ of S_{14}, where

$\beta = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13})$
$\beta_1 = (a_1, a_3, a_5, a_7, a_9, a_{11}, a_{13}, a_2, a_4, a_6, a_8, a_{10}, a_{12})$
$\beta_2 = (a_1, a_4, a_7, a_{10}, a_{13}, a_3, a_6, a_9, a_{12}, a_2, a_5, a_8, a_{11})$
$\beta_3 = (a_1, a_5, a_9, a_{13}, a_4, a_8, a_{12}, a_3, a_7, a_{11}, a_2, a_6, a_{10})$
$\beta_4 = (a_1, a_6, a_{11}, a_3, a_8, a_{13}, a_5, a_{10}, a_2, a_7, a_{12}, a_4, a_9)$
$\beta_5 = (a_1, a_7, a_{13}, a_6, a_{12}, a_5, a_{11}, a_4, a_{10}, a_3, a_9, a_2, a_8)$

and d is a positive integer . Thus,

(1)	$\beta^d = \beta \Longleftrightarrow d \equiv 1 \pmod{13}$
(2)	$\beta^d = \beta_1 \Longleftrightarrow d \equiv 2 \pmod{13}$
(3)	$\beta^d = \beta_2 \Longleftrightarrow d \equiv 3 \pmod{13}$
(4)	$\beta^d = \beta_3 \Longleftrightarrow d \equiv 4 \pmod{13}$
(5)	$\beta^d = \beta_4 \Longleftrightarrow d \equiv 5 \pmod{13}$
(6)	$\beta^d = \beta_5 \Longleftrightarrow d \equiv 6 \pmod{13}$
(7)	$\beta^d = \beta_5^{-1} \Longleftrightarrow d \equiv 7 \pmod{13}$
(8)	$\beta^d = \beta_4^{-1} \Longleftrightarrow d \equiv 8 \pmod{13}$
(9)	$\beta^d = \beta_3^{-1} \Longleftrightarrow d \equiv 9 \pmod{13}$
(10)	$\beta^d = \beta_2^{-1} \Longleftrightarrow d \equiv 10 \pmod{13}$
(11)	$\beta^d = \beta_1^{-1} \Longleftrightarrow d \equiv 11 \pmod{13}$
(12)	$\beta^d = \beta^{-1} \Longleftrightarrow d \equiv 12 \pmod{13}$

(ii) $A(\beta) = A(\beta^{-1})$, $A(\beta_1) = A(\beta_1^{-1})$, $A(\beta_2) = A(\beta_2^{-1})$, $A(\beta_3) = A(\beta_3^{-1})$, $A(\beta_4) = A(\beta_4^{-1})$ and $A(\beta_5) = A(\beta_5^{-1})$ [given that A_{14} is an ambivalent group]

(iii) $A(\beta) = A(\beta_2)$ [because, $\exists t = (a_2, a_4, a_{10})(a_6, a_3, a_7)(a_{11},$- *$t = (a_2, a_4, a_{10})(a_6, a_3, a_7)(a_{11}, a_5, a_{13})(a_8, a_9, a_{12}) \in A_{14}$ such that $t\beta t^{-1} = \beta_2$].*

(iv) $A(\beta) = A(\beta_3)$ [because, $\exists t = (a_6, a_8, a_3, a_9, a_7, a_{12})(a_{10},$- *$t = (a_6, a_8, a_3, a_9, a_7, a_{12})(a_{10}, a_{11}, a_2, a_5, a_4, a_{13}) \in A_{14}$ such that $t\beta t^{-1} = \beta_3$].*

(v) $A(\beta_1) = A(\beta_4)$ [because, $\exists t = (a_7, a_3, a_6)(a_5, a_{11}, a_{13})(a_9,$- *$t = (a_7, a_3, a_6)(a_5, a_{11}, a_{13})(a_9, a_8, a_{12})(a_4, a_2, a_{10}) \in A_{14}$ such that $t\beta_1 t^{-1} = \beta_4$].*

(vi) $A(\beta_1) = A(\beta_5)$ [because, $\exists t = (a_2, a_4, a_{10})(a_6, a_3, a_7)(a_{11},$- *$t = (a_2, a_4, a_{10})(a_6, a_3, a_7)(a_{11}, a_5, a_{13})(a_8, a_9, a_{12}) \in A_{14}$ such that $t\beta_1 t^{-1} = \beta_5$].*

Lemma 3.18. *Let $L = \{m \in N | m \equiv q \ (13)$ for some $q = 1,3,4,9,10,12\}$. If d is a positive integer such that $gcd(d,13) = 1$ and $\beta \in [1,13]$ of S_{14}, then the solutions of x^d -$\in A(\beta)$ in A_{14} are*

1. $A(\beta)$, if $d \in L$.
2. $A\left(\overset{\#}{\beta}\right)$, if $d \notin L$.

Proof. Given that $\beta \in [1,13] \cap H_{14}$, $[1,13]$ splits into two classes $A(\beta)$ and $A\left(\overset{\#}{\beta}\right)$ of A_{14}. Moreover, $\gcd(d,13) = 1$. So by (2.10), $[1,13] = A(\beta) \cup A\left(\overset{\#}{\beta}\right)$ is a solution set of $x^d \in [1,13] = A(\beta) \cup A\left(\overset{\#}{\beta}\right)$ in S_{14}. However, $A(\beta) \cap A\left(\overset{\#}{\beta}\right) = \phi$. Hence, for each $\pi \in [1,13] \Rightarrow \left(\pi \in A(\beta) \& \pi \notin A\left(\overset{\#}{\beta}\right)\right)$ or $(\pi \in A\left(\overset{\#}{\beta}\right) \& \pi \notin A(\beta))$.

(1) Assume $d \in L$. If $\pi \in A(\beta)$, then $(\pi \underset{A_{14}}{\approx} \beta)\pi$ is conjugate to β in A_{14}. However, $\pi^d \underset{A_{14}}{\approx} \pi$ (because $d \in L) \Rightarrow \pi^d \underset{A_{14}}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A\left(\overset{\#}{\beta}\right)$. If $\pi \in A\left(\overset{\#}{\beta}\right)$, then $(\pi \underset{A_{14}}{\approx} \overset{\#}{\beta})$. However, $\pi^d \underset{A_{14}}{\approx} \pi$ (because $d \in L) \Rightarrow \pi^d \underset{A_{14}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A\left(\overset{\#}{\beta}\right) \&$ $\pi^d \notin A(\beta)$. Then, the solution set of $x^d \in A(\beta)$ in A_{14} is $A(\beta)$.

(2) Assume $d \notin L$. If $\pi \in A(\beta)$, then $\left(\pi \underset{A_{14}}{\approx} \beta\right) \Rightarrow \overset{\#}{\pi} \underset{A_{14}}{\approx} \overset{\#}{\beta}$. However, $\pi^d \underset{A_{14}}{\approx} \overset{\#}{\pi}$ (because $d \notin L) \Rightarrow \pi^d \underset{A_{14}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A\left(\overset{\#}{\beta}\right) \&$ $\pi^d \notin A(\beta)$. If $\pi \in A\left(\overset{\#}{\beta}\right) \Rightarrow \left(\pi \underset{A_{14}}{\approx} \overset{\#}{\beta}\right) \Rightarrow \overset{\#}{\pi} \underset{A_{14}}{\approx} \beta$. However, $\pi^d \underset{A_{14}}{\approx} \overset{\#}{\pi}$ (because $d \notin L) \Rightarrow \pi^d \underset{A_{14}}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A\left(\overset{\#}{\beta}\right)$. Then, the solution set of $x^d \in A(\beta)$ in A_{14} is $A\left(\overset{\#}{\beta}\right)$.

Definition 3.19. Let $\beta = \gamma\lambda \in [3,11]$ of S_{14}, where $\lambda = (a_1, a_2, \text{-}$ $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11})$ *and* $\gamma = (b_1, b_2, b_3)$, *we define classes* $[3,11]\pm$ *of* A_{14} *by:*

$A(\beta) = [3,11]^+ = \{\mu \in [3,11] \mid \mu = t\beta t^{-1}$ for some $t \in A_{14}\}$ and $A\left(\overset{\#}{\beta}\right) = [3,11]^- = \{\mu \in [3,11] \mid \mu = t\overset{\#}{\beta}t^{-1}$ for some $t \in A_{14}\}$, where $\overset{\#}{\beta} = \gamma\lambda_1$ and $\lambda_1 = (a_1, a_3, a_5, a_7, a_9, a_{11}, a_2, a_4, a_6, a_8, a_{10})$

Remark 3.20.

(i) Let $\beta = \gamma\lambda \in [3,11]$ of S_{14}, where $\gamma = (b_1, b_2, b_3)$, $\lambda = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11})$,

$\lambda_1 = (a_1, a_3, a_5, a_7, a_9, a_{11}, a_2, a_4, a_6, a_8, a_{10})$,
$\lambda_2 = (a_1, a_4, a_7, a_{10}, a_2, a_5, a_8, a_{11}, a_3, a_6, a_9)$,
$\lambda_3 = (a_1, a_5, a_9, a_2, a_6, a_{10}, a_3, a_7, a_{11}, a_4, a_8)$,
$\lambda_4 = (a_1, a_6, a_{11}, a_5, a_{10}, a_4, a_9, a_3, a_8, a_2, a_7)$

and d is a positive integer. Thus,

(1)	$\beta^d = \beta \iff d \equiv 1 \pmod{33}$
(2)	$\beta^d = \gamma^{-1}\lambda_1 \iff d \equiv 2 \pmod{33}$
(3)	$\beta^d = \gamma\lambda_3 \iff d \equiv 4 \pmod{33}$
(4)	$\beta^d = \gamma^{-1}\lambda_4 \iff d \equiv 5 \pmod{33}$
(5)	$\beta^d = \gamma\lambda_3^{-1} \iff d \equiv 7 \pmod{33}$
(6)	$\beta^d = \gamma^{-1}\lambda_2^{-1} \iff d \equiv 8 \pmod{33}$
(7)	$\beta^d = \gamma\lambda^{-1} \iff d \equiv 10 \pmod{33}$
(8)	$\beta^d = \gamma\lambda_1 \iff d \equiv 13 \pmod{33}$
(9)	$\beta^d = \gamma^{-1}\lambda_2 \iff d \equiv 14 \pmod{33}$
(10)	$\beta^d = \gamma\lambda_4 \iff d \equiv 16 \pmod{33}$
(11)	$\beta^d = \gamma^{-1}\lambda_4^{-1} \iff d \equiv 17 \pmod{33}$
(12)	$\beta^d = \gamma\lambda_2 \iff d \equiv 14 \pmod{33}$
(13)	$\beta^d = \gamma^{-1}\lambda_1^{-1} \iff d \equiv 20 \pmod{33}$
(14)	$\beta^d = \gamma^{-1}\lambda \iff d \equiv 23 \pmod{33}$
(15)	$\beta^d = \gamma\lambda_2 \iff d \equiv 25 \pmod{33}$
(16)	$\beta^d = \gamma^{-1}\lambda_3 \iff d \equiv 26 \pmod{33}$
(17)	$\beta^d = \gamma\lambda_4^{-1} \iff d \equiv 28 \pmod{33}$
(18)	$\beta^d = \gamma^{-1}\lambda_3^{-1} \iff d \equiv 29 \pmod{33}$
(19)	$\beta^d = \gamma\lambda_1^{-1} \iff d \equiv 31 \pmod{33}$
(20)	$\beta^d = \gamma^{-1}\lambda^{-1} \iff d \equiv 32 \pmod{33}$

(ii) $A(\beta) = A(\beta^{-1})$, $A(\gamma^{-1}\lambda_1) = A(\gamma\lambda_1^{-1})$, $A(\gamma\lambda_3) = A(\gamma^{-1}\lambda_3^{-1})$, $A(\gamma^{-1}\lambda_4) = A(\gamma\lambda_4^{-1})$, $A(\gamma\lambda_3^{-1}) = A(\gamma^{-1}\lambda_3)$, $A(\gamma^{-1}\lambda_2^{-1}) = A(\gamma\lambda_2)$, $A(\gamma\lambda^{-1}) = A(\gamma^{-1}\lambda)$ [given that A_{14} is an ambivalent group]

(iii) $A(\beta) = A(\gamma\lambda_3)$ [because, $\exists t = (a_2, a_5, a_6, a_{10}, a_4)(a_3, a_9, \text{-}$ *$t = (a_2, a_5, a_6, a_{10}, a_4)(a_3, a_9, a_{11}, a_8, a_7) \in A_{14}$ such that $t\beta t^{-1} = \gamma\lambda_3$].*

(iv) $A(\beta) = A(\gamma\lambda_4)$ [because, $\exists t = (a_3, a_6, a_4, a_5, a_{10})(a_7, a_9, \text{-}$ *$t = (a_3, a_6, a_4, a_5, a_{10})(a_7, a_9, a_8, a_3, a_{11}) \in A_{14}$ such that $t\beta t^{-1} = \gamma\lambda_4$].*

(v) $A(\beta) = A(\gamma\lambda_2)$ [because, $\exists t = (a_6, a_5, a_2, a_4, a_{10})(a_9, a_3, \text{-}$ *$t = (a_6, a_5, a_2, a_4, a_{10})(a_9, a_3, a_7, a_8, a_{11}) \in A_{14}$ such that $t\beta t^{-1} = \gamma\lambda_2$].*

(vi) $A(\beta^{-1}) = A(\gamma^{-1}\lambda_1)$ [because, $\exists t = (a_8, a_9, a_7, a_{11}, a_3)(a_{10}, \text{-}$ *$t = (a_8, a_9, a_7, a_{11}, a_3)(a_{10}, a_5, a_4, a_6, a_2) \in A_{14}$ such that $t\beta t^{-1} = \gamma^{-1}\lambda_1$].*

Lemma 3.21. *Let $L = \{m \in N \mid m \equiv q \pmod{33}$ for some $q = 1, 2, 4, 8, 16, 17, 25, 29, 31, 32\}$. If d is a positive integer such that $\gcd(d,3) = 1$ & $\gcd(d,11) = 1$ and $\beta \in [3,11]$ of S_{14}, then the solutions of $x^d \in A(\beta)$ in A_{14} are*

1. $A(\beta)$, if $d \in L$.
2. $A\left(\overset{\#}{\beta}\right)$, if $d \notin L$.

Proof. Given that $\beta \in [3,11] \cap H_{14}$, $[3,11]$ splits into two classes $A(\beta)$ and $A\left(\overset{\#}{\beta}\right)$ of A_{14}. Moreover, $\gcd(d,3) = 1$ and $\gcd(d,11) = 1$. So by (2.10), $[3,11] = A(\beta) \cup A\left(\overset{\#}{\beta}\right)$ is a solution set of $x^d \in [3,11] = A(\beta) \cup A\left(\overset{\#}{\beta}\right)$ in S_{14}. However, $A(\beta) \cap A\left(\overset{\#}{\beta}\right) = \phi$. Hence, for each $\pi \in [3,11] \Rightarrow$ $\left(\pi \in A(\beta) \& \pi \notin A\left(\overset{\#}{\beta}\right)\right)$ or $\left(\pi \in A\left(\overset{\#}{\beta}\right) \& \pi \notin A(\beta)\right)$.

(1) Assume $d \in L$. If $\pi \in A(\beta)$, then $(\pi \underset{A_{14}}{\approx} \beta)\pi$ is conjugate to β in A_{14}. However, $\pi^d \underset{A_{14}}{\approx} \pi$ (because $d \in L) \Rightarrow \pi^d \underset{A_{14}}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A\left(\overset{\#}{\beta}\right)$. If $\pi \in A\left(\overset{\#}{\beta}\right)$, then $\left(\pi \underset{A_{14}}{\approx} \overset{\#}{\beta}\right)$. However, $\pi^d \underset{A_{14}}{\approx} \pi$ (because $d \in L) \Rightarrow \pi^d \underset{A_{14}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A\left(\overset{\#}{\beta}\right) \& \pi^d \notin A(\beta)$. Then, the solution set of $x^d \in A(\beta)$ in A_{14} is $A(\beta)$.

(2) Assume $d \notin L$. If $\pi \in A(\beta)$, then $(\pi \underset{A_{14}}{\approx} \beta) \Rightarrow \overset{\#}{\pi} \underset{A_{14}}{\approx} \overset{\#}{\beta}$. However, $\pi^d \underset{A_{14}}{\approx} \overset{\#}{\pi}$ (because $d \notin L) \Rightarrow \pi^d \underset{A_{14}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A\left(\overset{\#}{\beta}\right) \& \pi^d \notin A(\beta)$. If $\pi \in A\left(\overset{\#}{\beta}\right) \Rightarrow \left(\pi \underset{A_{14}}{\approx} \overset{\#}{\beta}\right) \Rightarrow \overset{\#}{\pi} \underset{A_{14}}{\approx} \beta$. However, $\pi^d \underset{A_{14}}{\approx} \overset{\#}{\pi}$ (because $d \notin L) \Rightarrow \pi^d \underset{A_{14}}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A\left(\overset{\#}{\beta}\right)$. Then, the solution set of $x^d \in A(\beta)$ in A_{14} is $A\left(\overset{\#}{\beta}\right)$. □

4. Concluding remarks

By the Cayley's theorem: Every finite group G is isomorphic to a subgroup of the symmetric group S_n, for some $n \geqslant 1$. Then we can discuss these propositions. Let $x^d = g$ be class equation in finite group G and assume that $f{:}G \cong A_n$, for some $n \in \theta$ and $f(g) \in H \cap C^\alpha$. The first question we are concerned with is: what is the possible value of d provided that there is no solution for $x^d = g$ in G? The second question we are concerned with is: what is the possible value of d provided that there is a solution for $x^d = g$ in G? and then we can find the solution and the number of the solution for $x^d = g$ in G by using Cayley's theorem and our theorems in this paper. In another direction, let G be a finite group, and $\pi_i(G) = \{g \in G|\ i$ the least positive integer number satisfies $g^i = 1\}$. If $|\pi_i(G)| = k_i$, then we write $\pi_i(G) = \{g_{i1}, g_{i2}, \ldots, g_{ik_i}\}$, and $\prod = \{\pi_i(G)\}_{i \geqslant 1}$. For each $g \in G$ and $g_{ij} \in \pi_i(G)$ we have $(gg_{ij}g^{-1})^i = 1$. By the Cayley's theorem we can suppose that $(f{:}G \cong S_n)$ or $(f{:}G \cong A_n)$. Also the questions can be summarized as follows:

1. Is $\prod = \{\pi_i(G)\}_{i \geqslant 1}$ collection set of conjugacy classes of G?
2. Is there some $i \geqslant 1$, such that $f^{-1}(C^\alpha) = \pi_i(G)$, for each C^α of A_n, where $(f{:}G \cong A_n)$?
3. Is there some $i \geqslant 1$, such that $f^{-1}(C^\alpha) = \pi_i(G)$, for each C^α of S_n, where $(f{:}G \cong S_n)$?
4. If $(G \cong S_n)$ and $p(n)$ is the number of partitions of, is $|\prod| = p(n)$?
5. If $(G \cong A_n)$ and A_n has m ambivalent conjugacy classes. Is it necessarily true that G has m ambivalent conjugacy classes?

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