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A Note on the Behavior of Karmarkar's Potential Function in Linear Programming

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Abstract: In this note we express Karmarkar's potential function in terms of the geometric mean of the decision variables of the linear programming problem, and obtain bounds on it. We also study the behaviour of the gradient and the hessian of the potential function at the center of the simplex and observe that the sum of all entries of the gradient and the hessian matrices at the center of the simplex are zero; and the center of the simplex is a saddle point for the potential function. Finally, we prove that the β-superlevel set of the function G(x) is a convex set.

Keywords: Potential function, saddle point, sublevel set and superlevel set.

1. Introduction

In 1984 N. Karmarkar of Bell Laboratory proposed a projective method to solve linear programming problems. From theoretical point of view it is a polynomial-time algorithm, which works on the interior of the feasible region. After seminal work of Karmarkar [7], thousands of paper appeared in literature on interior point methods (for details see [8] and various references cited in it). The most of these interior-point methods can be classified into three major categories, namely, 1) Affine Scaling Methods, 2) Potential Reduction Methods, and 3) The Central Path methods. The potential reduction methods make use of the potential function and generate sequence of points that minimizes the value of the potential function at each iteration. The details of the potential reduction methods can be found in Todd [9], Gonzaga [3] and Singh and Singh [8]. In this note our objective is to study the behavior of the primal-potential function used in Karmarkar [7]. The primal-potential functions have been used by some other authors also, for example Gonzaga [3]. The details of the other types of potential functions like primal-dual potential functions, the symmetric primal-dual potential functions etc. can be found in Todd [7] and Singh and Singh [8]. The potential reduction methods use potential function to optimize the objective function of the linear programming problems. Karmarkar's potential function is used to measure the progress at each iteration, analyze the convergence and facilitate the complexity analysis of the algorithm. It is expected that the study of the behavior of the potential function may be useful in developing and analyzing the efficient algorithms for the linear programming problems, based on the reduction of the potential functions. The original algorithm of Karmarkar [7] considers a linear programming problem in canonical form:

$$\begin{aligned} &Minimize\ c^T x \\ &s.t.\ Ax = b \\ &e^T x = 1 \\ &x \geq 0, \end{aligned} \tag{1}$$

where $c \in Z^n, A \in Z^{m\times n}, and\ e^T = (1,1,\ldots,1),$ which can be obtained from a standard form linear programming problem

$$\begin{aligned} &Minimize\ c^T x \\ &s.t.\ Ax = b \\ &x \geq 0, \end{aligned} \tag{2}$$

by using a projective transformation. The targeted minimum value of the objective function $c^T x$ is zero, which in turn implies that c does not depend linearly on vector of all ones e = $(1,1,\ldots,1)^T$. For transforming a linear programming in standard form to canonical form Karmarkar [7] has used the following projective transformation.

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$$x' = T(x) = \frac{X^{-1}x}{e^T X^{-1}x} \tag{3}$$

and whose inverse transformation is given by

$$x = T^{-1}(x') = \frac{Xx'}{e^T Xx'},$$

where $X = diag(x) = \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n \end{pmatrix}$ a diagonal matrix is whose diagonal elements are the components of vector $x \in R^n$.

1.1 The potential Function

We know that linear functions are not invariant under projective transformations, but the ratio of two linear functions can be transformed to the ratio of the two linear functions. Keeping this fact in mind Karmarkar [7] has introduced a potential function $f : R_+^n \to R$ as a ratio of two linear functions as follows:

$$f(x) = \sum_{j=1}^{n} \ln \frac{c^T x}{x_j} = n \ln c^T x - \sum_{j=1}^{n} \ln x_j \quad ,(4)$$

where $R_+^n = \{x \in R^n : x_i > 0\}$ is called positive orthant of R^n. It is easy to see that Karmarkar's potential function $f(x)$ is a homogeneous function of degree zero. The potential function consists of two terms: $n \ln c^T x$ and $\sum_{j=1}^{n} \ln x_j$. The second term of the potential function is the logarithmic barrier function, defined on the interior of positive orthant of R^n, which compels the feasible solution to remain away from the boundary of the feasible region. This potential function is used to measure the progress of the algorithm. We expect a certain decrease δ in the potential function at each iteration of the algorithm. If we don't observe the expected improvement, i.e., $f\left(x^{(k+1)}\right) > f\left(x^{(k)}\right) - \delta$ then we stop, and conclude that the minimum value of the objective function $c^T x$ is strictly positive. This situation corresponds to the case that the original problem does not have finite optimum i.e., it is either *infeasible* or *unbounded*.

In this note we study the behavior of the Karmarkar's potential function at the center of the simplex and the level set properties of the function $G(x) = \left(\prod_{j=1}^{n} x_j\right)^{1/n}$ used in the potential function. Imai [6] has also studied the convexity of the multiplicative version of the Karmarkar's potential function and proved that the Karmarkar's potential function is not necessarily convex in general, but is strictly convex when the feasible region is bounded. As the Karmarkar's potential function works on the simplex, a bounded set, so it is convex function. Karmarkar's algorithm can be viewed as a gradient projection method for minimizing potential function $f(x) = \sum_{j=1}^{n} \ln \frac{c^T x}{x_j}$ in projected space, so the study of its properties may be useful in designing the efficient potential reduction algorithms for the other classes of problems like semidefinite programming (SDP) and second order cone programming (SOCP).

In section 2.1 of this note we express the Karmarkar's potential function in the terms of the geometric mean $G(x)$ of the decision variables $x_1, x_2, \ldots, x_n$ and obtain the lower and the upper bounds on it, in the terms of the matrices used to define projective transformations of the algorithm. In section 2.2 we study the behavior of the gradient and the hessian of the potential function $f(x)$) at the center $a_0 = (1/n, 1/n, \ldots, 1/n)$ of the simplex $S = \{x : x_i > 0, e^T x = 1\}$ and prove that: a) $e^T[\nabla^2 f(a_0)] = 0$; b) $e^T[\nabla^2 f(a_0)]e = 0$ and c) the center of the simplex a_0 is a saddle point of the potential function $f(x)$. Finally in theorem 2 we study the problems related to the level set and convexity of function $G(x)$ and prove that the β-superlevel set of function $G(x)$ is a convex set.

2. PROPERTIES OF THE KARMARKAR'S POTENTIAL FUNCTION

2.1 Bounds on potential Function

The Karmarkar's Potential function $f(x) = \sum_{j=1}^{n} \ln \frac{c^T x}{x_j}$ can be written as $f(x) = \ln \left[\frac{(c^T x)}{G(x)}\right]^n$, where $G(x) = \left(\prod_{j=1}^{n} x_j\right)^{1/n}$ is the geometric mean of

$x_1, x_2, \ldots, x_n$. If $A(x)$ and $H(x)$ denote the arithmetic and the harmonic means respectively then

$A(x) = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{e^T Xe}{n}$ and $H(x) = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}} = \frac{n}{e^T X^{-1} e}$.

Since, $A(x) \geq G(x) \geq H(x)$ we have

$$\ln\left[\frac{(c^T x)}{A(x)}\right]^n \leq \ln\left[\frac{(c^T x)}{G(x)}\right]^n \leq \ln\left[\frac{(c^T x)}{H(x)}\right]^n$$

i.e. $\ln\left[\frac{(c^T x)}{A(x)}\right]^n \leq f(x) \leq \ln\left[\frac{(c^T x)}{H(x)}\right]^n$

i.e., $\ln\left[\frac{n(c^T x)}{e^T Xe}\right]^n \leq f(x) \leq \ln\left[\frac{(c^T x)e^T X^{-1} e}{n}\right]^n$ (5)

2.2 Behavior of the Gradient and the Hessian of the Karmarkar's Potential Function at the Center of the Simplex

At each iteration of Karmarkar's algorithm the current iterate $x^{(k)}$ is mapped to the center $a_0 = (1/n, 1/n, \ldots, 1/n)$ of the simplex S. In this subsection we study the behavior of the Karmarkar's potential function at the center of the simplex.

Theorem 1 let $f(x) = \sum_{j=1}^{n} \ln \frac{c^T x}{x_j}$ *be the Karmarkar's potential function and* a_0 *be the center of the simplex,*

a) $e^T[\nabla f(a_0)] = 0$

b) $e^T[\nabla^2 f(a_0)]e = 0$ *, and*

c) The center of the simplex is a saddle point for the potential function $f(x)$.

Proof. a) We have $\nabla f(x) = \frac{nc}{c^T x} - X^{-1}e,$ where X is defined as before and e denotes the vector of ones.

We have $a_0 = \frac{1}{n}e,$

i.e., $[\nabla f(a_0)] = \frac{n^2 c}{c^T e} - ne$

i.e., $e^T[\nabla f(a_0)] = e^T\left[\frac{n^2 c}{c^T e} - ne\right] = n^2 - n.n = 0$

b) We have $\nabla^2 f(x) = X^{-2} - \frac{n}{\left(c^T x\right)^2} - X^{-1}e,$

i.e., $\nabla^2 f(a_0) = n^2 I - \frac{n^3}{\left(c^T e\right)^2} cc^T$, where I is the identity matrix.

i.e., $e^T\left[\nabla^2 f(a_0)\right]e = n^2.n - n^3 = 0.$

c) We have $\nabla^2 f(a_0) = n^2 I - \frac{n^3}{\left(c^T e\right)^2} cc^T$.

The eigenvalues of $\nabla^2 f(a_0)$ are of the form $n^2 - \frac{n^3}{(c^T e)^2}\lambda,$ where λ runs through the eigenvalues of cc^T . The eigenvalues of cc^T being 0 with multiplicity $(n-1)$ and $\|c\|$ .The eigenvalues of $\nabla^2 f(a_0)$ are n^2 and $n^2 - \frac{n^3\|c\|^2}{(c^T e)^2}$. obviously $n^2 > 0$ and one can easily find a vector c so that the eigenvalue is negative. In fact due to Cauchy-Schwartz inequality this hold for any vector c which is linearly independent of e.

We have,

$$(c^T e)^2 < \|c\|^2 \|e\|^2 = n\|c\|^2$$

Now, $n^2 - \frac{n^3\|c\|^2}{(c^T e)^2} < n^2 - \frac{n^3\|c\|^2}{n\|c\|} = 0$.

Thus we see that $\nabla^2 f(a_0)$ has both positive and negative eigenvalues, so the center of the simplex is a saddle point for Karmarkar's potential function.

Theorem 2 The β-superlevel set of the function G(x), used in the definition of Karmarkar's potential function is a convex set.

Proof. We have $G(x)=\left(\prod_{j=1}^{n} x_j\right)^{1/n}$,

$$\frac{\partial^2 G(x)}{\partial {x_k}^2}=\frac{-(n-1)}{n^2}\frac{\left(\prod_{j=1}^{n} x_j\right)^{1/n}}{{x_k}^2}=\frac{-(n-1)}{n^2}\frac{G(x)}{{x_k}^2},\quad k=1,2,\dots,n \quad and$$

$$\frac{\partial^2 G(x)}{\partial x_k \partial x_l}=\frac{\left(\prod_{j=1}^{n} x_j\right)^{1/n}}{n^2 x_k x_l}=\frac{G(x)}{n^2 x_k x_l},\quad if \quad k\neq l\,.$$

$$\nabla^2 G(x)=\begin{pmatrix} \frac{\partial^2\Omega}{\partial {x_1}^2} & \frac{\partial^2\Omega}{\partial x_1\partial x_2} & \cdots & \frac{\partial^2\Omega}{\partial x_1\partial x_n} \\ \frac{\partial^2\Omega}{\partial x_2\partial x_1} & \frac{\partial^2\Omega}{\partial {x_2}^2} & \cdots & \frac{\partial^2\Omega}{\partial x_2\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2\Omega}{\partial x_n\partial x_1} & \frac{\partial^2\Omega}{\partial x_n\partial x_2} & \cdots & \frac{\partial^2\Omega}{\partial {x_n}^2} \end{pmatrix} = -\frac{G}{n^2}\begin{pmatrix} \frac{n}{{x_1}^2}-\frac{1}{{x_1}^2} & -\frac{1}{x_1x_2} & \cdots & -\frac{1}{x_1x_n} \\ -\frac{1}{x_2x_1} & \frac{n}{{x_2}^2}-\frac{1}{{x_2}^2} & \cdots & -\frac{1}{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{x_nx_1} & -\frac{1}{x_nx_2} & \cdots & \frac{n}{{x_n}^2}-\frac{1}{{x_n}^2} \end{pmatrix}$$

$$=-\frac{G}{n^2}\left[n\begin{pmatrix} \frac{1}{{x_1}^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{{x_2}^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{{x_n}^2} \end{pmatrix} - \begin{pmatrix} \frac{1}{{x_1}^2} & \frac{1}{x_1x_2} & \cdots & \frac{1}{x_1x_n} \\ \frac{1}{x_2x_1} & \frac{1}{{x_2}^2} & \cdots & \frac{1}{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_nx_1} & \frac{1}{x_nx_2} & \cdots & \frac{1}{{x_n}^2} \end{pmatrix} \right]$$

This implies for all $u\in R^n$, we have

$$u^T\left[\nabla^2 G(x)\right]u=-\frac{G}{n^2}\left[n\sum_{i=1}^{n}\left(\frac{u_i}{x_i}\right)^2-\left(\sum_{i=1}^{n}\frac{u_i}{x_i}\right)^2\right] \quad (6)$$

For any two vector r and s in R^n, it follows from Cauchy-Schwartz inequality that

$$(r^T s)^2 \le (r^T r)(s^T s).$$

In particular, if we take

$$r = e = (1,1,\ldots,1) \text{ and } s = \left(\frac{u_1}{x_1}, \frac{u_2}{x_2}, \ldots, \frac{u_n}{x_n}\right) \text{ then}$$

$$\left(\sum_{i=1}^{n} \frac{u_i}{x_i}\right)^2 \le n \sum_{i=1}^{n} \left(\frac{u_i}{x_i}\right)^2 \text{ this implies}$$

$$n \sum_{i=1}^{n} \left(\frac{u_i}{x_i}\right)^2 - \left(\sum_{i=1}^{n} \frac{u_i}{x_i}\right)^2 \ge 0$$

$$\Rightarrow u^T \left[\nabla^2 G(x)\right] u = -\frac{G}{n^2}\left[n \sum_{i=1}^{n} \left(\frac{u_i}{x_i}\right)^2 - \left(\sum_{i=1}^{n} \frac{u_i}{x_i}\right)^2\right] \le 0.$$

This means the hessian matrix $\nabla^2 G(x)$ of $G(x)$ is negative definite, so $G(x)$ is a concave function. Hence it follows that the $\beta - \sup erlevel\ set\ of\ G(x)$, i.e.,

$$L_\beta = \left\{ x : x > 0, \left(\prod_{j=1}^{n} x_j\right)^{1/n} \ge 0 \right\} \text{ is a convex set.}$$

3. Concluding Remarks

The linear programming models are one of the most frequently used models to solve a large variety of real-life problems in operations research. The application of linear programming in agriculture, telecommunication, transportation problems, maximization of profit and minimization of production costs etc. are well known in literature. It remains one of many operational research techniques used by armed forces worldwide. Before Karmarkar [7], developed an efficient interior-point-method to solve linear programming problems, large linear programming problems were not solvable. In these days a linear programming problem with millions of variables and equations can be solved using various interior-point-methods developed after seminal work of Karmarkar [7]. As linear programming models are used in various sectors of economy and business, developing efficient algorithms to solve linear programming problems using various variants of potential-reduction methods will be extremely useful in decision making for the problems related to economy, business and industries. Starting from Karmarkar [7] several types of potential functions have been proposed including primal-potential functions, dual-potential functions, primal-dual potential functions and symmetric primal-dual-potential functions. Quite recently, the Semidefinite Programming (SDP) and the Second Order Cone Programming (SOCP) have appeared in literature as a generalization of linear programming. All the interior-point-methods including potential-reduction methods are being used to develop efficient algorithms for solving SDP and SOCP (further details of potential-reduction methods and some issues related to efficient implementation of SDP algorithms can be found in [4,10]). It is expected that the various results developed in this paper will be helpful in designing and developing efficient algorithms not only for linear programming but for the SDP and the SOCP also.

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