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Generalized Order Statistics from $q-$Exponential Type- I Distribution and its Characterization

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Abstract: This article is concerned with $q-$ exponential type-1 distribution. Recurrence relations for single and product moments of generalized order statistics have been derived from $q-$ exponential type-1 distribution. Single and product moments of ordinary order statistics and upper k records cases have been discussed as a special case from generalized order statistics.

Keywords: Generalized order statistics, Order statistics, Record values, Single and product moments, Recurrence relations, $q-$ exponential type-1 distribution and characterization.

1. Introduction

Kamps (1995) introduced the concept of generalized order statistics (gos) as follows:
Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (iid) random variable (rv) with the df $F(x)$ and the pdf $f(x)$. Let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, 2, \ldots, n-1\}$. Then $X(r, n, \tilde{m}, k)$, $r = 1, 2, \ldots, n$ are called (gos) if their joint pdf is given by

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\right)[1-F(x_n)]^{k-1} f(x_n) \tag{1.1}$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1)$ of $\Re^n$.
The joint density of the first r - gos is given by

$$f_{X(1,n,\tilde{m},k),\ldots,X(r,n,\tilde{m},k)}(x_1, x_2, \ldots, x_r)$$
$$= C_{r-1}\left(\prod_{i=1}^{r-1}[\bar{F}(x_i)]^{m_i} f(x_i)\right)[\bar{F}(x_r)]^{k+n-r+M_r-1} f(x_r) \tag{1.2}$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1)$.
Then it is called generalized order statistics of a sample from distribution with df $F(x)$.
The pdf of r^{th} $m-gos$ is given by [Kamps, 1995]:

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}[F(x)] \tag{1.3}$$

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, the r^{th} and s^{th} $m-gos$, $1 \leq r < s \leq n$, is

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$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^{m}g_{m}^{r-1}[F(x)]$$
$$\times[h_{m}(F(y))-h_{m}(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_{s}-1}f(x)f(y),\qquad \alpha\le x<y\le\beta \tag{1.4}$$

where

$$C_{r-1}=\prod_{i=1}^{r}\gamma_{i},\quad \gamma_{i}=k+(n-i)(m+1),$$

$$h_{m}(x)=\begin{cases}-\dfrac{1}{m+1}(1-x)^{m+1}, & m\ne-1\\ -\log(1-x), & m=-1\end{cases}$$

and

$$g_{m}(x)=\int_{0}^{x}(1-t)^{m}dt=h_{m}(x)-h_{m}(0),\qquad x\in[0,1).$$

We define the $q-$ exponential distribution is a generalization of the exponential distribution. The main reason for introducing $q-$ exponential model is the switching property of the exponential form to corresponding binomial expansion. We refer the reader to Seetha and Thomas (2012) for a comprehensive study on the properties of $q-$ exponential distribution

$$\lim_{q\to1}[1+(q-1)z]^{-\frac{1}{q-1}}=e^{-z},\ 1<q<2$$

The main properties of the $q-$ exponential distribution as follows,

(1) Exponential distribution is a special case
(2) It has equi- dispersed data via shape parameter
(3) It allows for non- constant hazard rates

A random variable X is said to have $q-$ exponential type-1 distribution $(0<q<1)$ if its pdf is given by

$$f(x)=\upsilon(2-q)[1-(1-q)(\upsilon x)]^{\frac{1}{1-q}},\ q<1,\ 0\le x\le\beta,\upsilon>0 \tag{1.5}$$

where $\beta=\dfrac{1}{\upsilon(1-q)}$

and the corresponding df is

$$\bar{F}(x)=[1-(1-q)(\upsilon x)]^{\frac{2-q}{1-q}} \tag{1.6}$$

Therefore, in view of (1.5) and (1.6), we have

$$\bar{F}(x)=\frac{[1-(1-q)(\upsilon x)]}{\upsilon(2-q)}f(x) \tag{1.7}$$

Kamps (1998) investigated the importance of recurrence relations of order statistics in characterization. Recurrence relations for moments of order statistics and upper k-records were investigated, among others, by Khan *et al*. (1983a, 1983b), Grudzien and Szynal (1997) and Pawlas and Szynal (1998, 1999) among others.

In this paper, we are concerned with the generalized order statistics from $q-$ exponential type-1 distribution. Sections 2 and 3 give the recurrence relations for single and product moments of generalized order statistics. Section 4 is based on the characterization result.

2. RECURRENCE RELATIONS FOR SINGLE MOMENTS

THEOREM 2.1: For the $q-$ exponential type-1 distribution given (1.5) and $n \in N, m \in R$, $2 \le r \le n$

$$E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)] =$$
$$\frac{j}{\gamma_r \upsilon(2-q)}\left\{E[X^{j-1}(r,n,m,k)] - \nu(1-q)\ E[X^j(r,n,m,k)]\right\} \qquad (2.1)$$

PROOF: From (1.3), we have

$$E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!}\int_0^\beta x^j[\bar{F}(x)]^{\gamma_r-1} f(x)\, g_m^{r-1}(F(x))dx \qquad (2.2)$$

Integrating by parts taking $[\bar{F}(x)]^{\gamma_r-1} f(x)$ as the part to be integrated, we get

$$E[X^j(r,n,m,k)] = E[X^j(r-1,n,m,k)] + \frac{jC_{r-1}}{\gamma_r (r-1)!}\int_0^\beta x^{j-1}[\bar{F}(x)]^{\gamma_r} g_m^{r-1}(F(x))dx$$

The constant of integration vanishes since the integral considered in (2.2) is a definite integral, on using (1.7), we obtain

$$E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)] = \frac{j}{\gamma_r\, \upsilon(2-q)}\left\{E[X^{j-1}(r,n,m,k)] - \nu(1-q)\ E[X^j(r,n,m,k)]\right\}$$

and hence the Theorem

REMARK 2.1: Setting $m=0, k=1$ in the Theorem 2.1, we obtain the recurrence relations for the single moments of order statistics of the $q-$ exponential type-1 distribution in the form

$$E[X^j{}_{r:n}] - E[X^j{}_{r-1:n}] = \frac{j}{\upsilon(2-q)(n-r+1)}\left\{E[X^{j-1}{}_{r:n}] - \nu(1-q)\ E[X^j{}_{r:n}]\right\}$$

REMARK 2.2: Setting $m=-1, k=1$ in the Theorem 2.1, we get the recurrence relations for the single moments of upper $k-$ record of the $q-$ exponential type-1 distribution in the form

$$E[X^j{}_{U(r)}]^k - E[X^j{}_{U(r-1)}]^k = \frac{j}{\upsilon(2-q)k}\left\{E[X^{j-1}{}_{U(r)}]^k - \nu(1-q)\ E[X^j{}_{U(r)}]^k\right\}$$

3. RECURRENCE RELATIONS FOR PRODUCT MOMENTS

THEOREM 3.1: For the $q-$ exponential type-1distribution given (1.5) and $n \in N, m \in R$, $1 \le r \le s \le n-1$

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)] - E[X^i(r,n,m,k)\ X^j(s-1,n,m,k)] =$$
$$\frac{j}{\gamma_s\, \upsilon(2-q)}\left\{E[X^i(r,n,m,k)\ X^{j-1}(s,n,m,k)] - \upsilon(1-q)E[X^i(r,n,m,k)\ X^j(s,n,m,k)]\right\} \qquad (3.1)$$

PROOF: From (1.4), we have

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\beta x^i[\bar{F}(x)]^m f(x)\, g_m^{r-1}(F(x))\ I(x)dx \qquad (3.2)$$

where

$$I(x) = \int_x^\beta y^j[\bar{F}(x)]^{\gamma_s-1}[h_m(F(y) - h_m(F(x)]^{s-r-1} f(y)\, dy$$

Solving the integral in $I(x)$ by parts and substituting the resulting expression in (3.2), we get

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)] - E[X^i(r,n,m,k)\ X^j(s-1,n,m,k)] =$$
$$\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\beta\int_x^\beta x^i y^{j-1}[\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)$$
$$\times[h_m(F(y)-h_m(F(x)]^{s-r-1}[F(y)]^{\gamma_s}\,dy\,dx$$

The constant of integration vanishes since the integral in $I(x)$ is definite integral. On using relation (1.7), we obtain

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)] - E[X^i(r,n,m,k)\ X^j(s-1,n,m,k)] =$$
$$\frac{jC_{s-1}}{\gamma_s\upsilon(2-q)(r-1)!(s-r-1)!}\left\{\int_0^\beta\int_x^\beta x^i y^{j-1}[\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)\right.$$
$$\times[h_m(F(y)-h_m(F(x)]^{s-r-1}[F(y)]^{\gamma_s-1} f(y)\,dy\,dx$$
$$\left.-\upsilon(1-q)\int_0^\beta\int_x^\beta x^i y^j[\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)[h_m(F(y)-h_m(F(x)]^{s-r-1}[F(y)]^{\gamma_s-1} f(y)\,dy\,dx\right\}$$

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)] - E[X^i(r,n,m,k)\ X^j(s-1,n,m,k)] =$$
$$\frac{j}{\gamma_s\,\upsilon(2-q)}\left\{E[X^i(r,n,m,k)\ X^{j-1}(s,n,m,k)] - \upsilon(1-q)E[X^i(r,n,m,k)\ X^j(s,n,m,k)]\right\}$$

and hence the Theorem

REMARK 3.1: Setting $m=0, k=1$ in the Theorem 3.1, we obtain the recurrence relations for the product moments of order statistics of the $q-$ exponential type-1 distribution in the form

$$E[X_{r,s:n}^{i\ j}] - E[X_{r,s-1:n}^{i\ j}] = \frac{j}{\upsilon(2-q)(n-s+1)}\left\{E[X_{r,s:n}^{i\ j-1}] - \nu(1-q)\ E[X_{r,s:n}^{i\ j}]\right\}$$

REMARK 3.2: Setting $m=-1, k=1$ in the Theorem 3.1, we get the recurrence relations for the product moments of upper $k-$ record of the $q-$ exponential type-1 distribution in the form

$$E[X^i{}_{U(r)}\ X^j{}_{U(s)}]^k - E[X^i{}_{U(r)}\ X^j{}_{U(s-1)}]^k = \frac{j}{\upsilon(2-q)k}\left\{E[X^i{}_{U(r)} X^{j-1}{}_{U(s)}]^k - \nu(1-q)\ E[X^i{}_{U(r)}\ X^j{}_{U(s)}]^k\right\}$$

4. CHARACTERIZATION

THEOREM 4.1: Let X be a non-negative random variable having absolutely continuous distribution $F(x)$ with $F(0)=0$ and $0<F(x)<1$, for all $x>0$

$$E[X^j(r,n,m,k)] = E[X^j(r-1,n,m,k)]$$
$$+\frac{j}{\gamma_r\ \upsilon(2-q)}E[X^{j-1}(r,n,m,k)] - \frac{j(1-q)}{\gamma_r(2-q)}\ E[X^j(r,n,m,k)] \quad (4.1)$$

if and only if

$$\bar{F}(x) = [1-(1-q)(\upsilon x)]^{\frac{2-q}{1-q}}$$

Proof: The necessary part follows immediately from equation (2.1). On the other hand if the recurrence relation in equation (4.1) is satisfied, then on using equation (1.3), we have

$$\frac{C_{r-1}}{(r-1)!}\int_0^\beta x^j[\bar{F}(x)]^{\gamma_r-1} f(x)\,g_m^{r-1}(F(x))dx = \frac{(r-1)C_{r-1}}{\gamma_r\,(r-1)!}\int_0^\beta x^j[\bar{F}(x)]^{\gamma_r+m} f(x)\,g_m^{r-2}(F(x))dx$$
$$+\frac{j\,C_{r-1}}{\gamma_r\,\upsilon(2-q)(r-1)!}\int_0^\beta x^{j-1}[\bar{F}(x)]^{\gamma_r-1} f(x)\,g_m^{r-1}(F(x))\,dx$$
$$-\frac{j\,C_{r-1}\,(1-q)}{\gamma_r\;(r-1)!(2-q)}\int_0^\beta x^j[\bar{F}(x)]^{\gamma_r-1} f(x)\,g_m^{r-1}(F(x))\,dx \qquad (4.2)$$

Integrating the first integral on the right hand side of equation (4.2), by parts, we get

$$\frac{C_{r-1}}{(r-1)!}\int_0^\beta x^j[\bar{F}(x)]^{\gamma_r-1} f(x)\,g_m^{r-1}(F(x))dx = -\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\beta x^{j-1}[\bar{F}(x)]^{\gamma_r} f(x)\,g_m^{r-1}(F(x))dx$$
$$+\frac{C_{r-1}}{(r-1)!}\int_0^\beta x^j[\bar{F}(x)]^{\gamma_r-1} f(x)\,g_m^{r-1}(F(x))dx$$
$$+\frac{j\,C_{r-1}}{\gamma_r\,\upsilon(2-q)(r-1)!}\int_0^\beta x^{j-1}[\bar{F}(x)]^{\gamma_r-1} f(x)\,g_m^{r-1}(F(x))\,dx$$
$$-\frac{j\,C_{r-1}\,(1-q)}{\gamma_r\;(r-1)!(2-q)}\int_0^\beta x^j[\bar{F}(x)]^{\gamma_r-1} f(x)\,g_m^{r-1}(F(x))\,dx$$

which reduces to

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\beta x^{j-1}[\bar{F}(x)]^{\gamma_r-1} g_m^{r-1}(F(x))\left[\bar{F}(x)-\frac{1}{\upsilon(2-q)}\,f(x)+x\frac{(1-q)}{(2-q)}\,f(x)\right]dx=0\,. \qquad (4.3)$$

Now applying a generalization of the Muntz- Szasaz Theorem (Hawang and Lin, 1984) to equation (4.3), we get

$$\frac{f(x)}{\bar{F}(x)}=\frac{\upsilon(2-q)}{[1-(1-q)\upsilon x]}$$

which proves that

$$\bar{F}(x)=[1-(1-q)\upsilon x]^{\frac{2-q}{1-q}}$$

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