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Mean Absolute Deviation: Analysis and Applications

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Abstract: One way analysis of mean absolute deviation (ANOMAD) about mean and median is derived where the total sum of absolute deviation is partition into exact between sum of absolute deviation and within sum of absolute deviation. Because of the presence of absolute function in mean absolute deviation, the middle term does not exist where it is included in each term. Therefore, the exact partitions are derived using the idea of creating error terms from the same type and take it away from each term to obtain the exact partitions. ANOMAD has advantages: does not square data and offers meaningful measure of dispersion. However, the variance-gamma distribution is used to fit the sampling distributions of between sum of absolute and within sum of absolute. Consequently, two tests of equal medians and means are introduced under the assumption of Laplace distribution. Moreover, two measures of effect sizes are re-defined and studied in terms of ANOMAD.

Keywords: ANOVA, Effect sizes, Laplace distribution, MAD, Variance-gamma distribution.

1. INTRODUCTION

Mean absolute deviation (MAD) is a natural scale parameter of the Laplace distribution and offers a direct and meaningful measure of the dispersion of a random variable. It has been used as an alternative to the standard deviation where the standard deviation is motivated from optimality results in independent random sampling from the normal distribution, an analysis dating back to Fisher; see, [18]. However, the mean absolute deviation may be more appropriate in case of departures from normality or in presence of outliers; see, [19] and [9]. It may also offer certain pedagogical advantages; see, [1] and [7]. For extensive discussion and comparisons; see [15] and [6] and the references therein.

The population of MAD about mean and median is defined as

$$\Delta_\mu = E|Y-\mu|$$

and

$$\Delta_\nu = E|Y-\nu|$$

where $\mu = E(Y)$ and $\nu = Med(Y)$ are the population mean and median, respectively.

The sample MAD is defined as

$$d_{\bar{x}} = \frac{1}{n}\sum_{i=1}^{n}|Y_i-\bar{Y}| \quad \text{and} \quad d_{\tilde{x}} = \frac{1}{n}\sum_{i=1}^{n}|Y_i-\tilde{Y}|$$

where $\bar{Y}$ and $\tilde{Y}$ are the sample mean and median, respectively.

A random variable has a Laplace distribution with location parameter $-\infty<\mu<\infty$ and scale $\Delta>0$ if its probability density function is

$$f(y) = \frac{1}{2\Delta}e^{-\frac{|y-\mu|}{\Delta}}, \quad -\infty<y<\infty$$

The Laplace distribution has

$$E(Y)=\mu, \qquad V(Y)=2\Delta^2 \quad \text{and} \quad MAD(Y)=\Delta$$

The probability density function of the Laplace distribution is also reminiscent of the normal distribution; whereas the normal distribution is expressed in terms of the squared difference from the mean while the Laplace density is expressed in terms of the absolute difference from the mean or median. Consequently, the Laplace distribution has flatter tails than the normal distribution and recently applied to many fields, engineering, financial, inventory management and quality control; see, [10].

Because of the presence of absolute function in MAD, the middle term does not exist where it is included inside each term.

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Therefore, an exact partition of the total sum of absolute deviation (TSA) about mean and median into exact between sum of absolute (EBSA) and exact within sum of absolute (EWSA) is derived by using the idea of creating error term of the same type (between and within) and take it away from each term to obtain EBSA and EWSA.

Since the MAD is a natural parameter of the Laplace distribution the sampling distributions of the EBSA and EWSA are studied under the assumption of Laplace distribution using variance-gamma distribution (generalized Laplace distribution). Consequently, an analysis of mean absolute deviation (ANOMAD) is introduced to test for equal population means and medians. Moreover, two measures of effect sizes are re-expressed and studied in terms of TSA, EBSA and EWSA.

Representation of MAD as a covariance is presented in Section 2. The exact partitions of TSA into EBSA and EWSA are derived in Section 3. The sampling distributions of EBSA and EWSA, and tests for equal means and median are presented in Section 4. Section 5 is devoted to conclusion.

2. Representation of MAD as a Covariance

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a continuous distribution with, density function $f(y)$, quantile function $y(F) = F^{-1}(y) = Q(F)$, $0 < F < 1$, cumulative distribution function $F(y) = F$. Let over indicator functions are

$$I_O(Y,\mu) = I_{O\mu} = \begin{cases} 1, & Y > \mu \\ 0, & \text{else} \end{cases}$$

and (1)

$$I_O(Y,\nu) = I_{O\nu} = \begin{cases} 1, & Y > \nu \\ 0, & \text{else} \end{cases}$$

[8] and [4] used the general dispersion function that defined by [14] as

$$\begin{aligned} \Delta_Y(a) &= E|Y-a| \\ &= a\left[2F_y(a) - 1\right] + E(Y) \\ &\quad - 2\int YI_{Y<a}\,dP \end{aligned} \tag{2}$$

to re-define the population MAD about mean and median in terms of over indicator functions as

$$\begin{aligned} \Delta_Y(\mu) = D_\mu &= E|Y-\mu| \\ &= E\{2[I_{0\mu} - E(I_{O\mu})]Y\}, \end{aligned} \tag{3}$$

$$\Delta_Y(\nu) = D_\nu = E|Y-\nu| = E[(2I_{O\nu} - 1)Y]$$

Consequently, the population MAD is re-defined as twice the covariance between the random variable Y and its indicator function as

$$\begin{aligned} \Delta_\mu &= 2\left[E(YI_{O\mu}) - E(Y)E(I_{O\mu})\right] \\ &= 2Cov(Y, I_{O\mu}), \end{aligned}$$

and (4)

$$\begin{aligned} \Delta_\nu &= 2E(YI_{O\nu}) - 2E(Y)E(I_{O\nu}) \\ &= 2Cov(Y, I_{O\nu}) \end{aligned}$$

Note that $E(I_{o\mu}) = F(\mu)$, $V(I_{o\mu}) = F(\mu)(1-F(\mu))$, $E(I_{o\nu}) = 0.5$ and $V(I_{o\nu}) = 0.25$.

3. Exact MAD Partitions about Mean and Median

Assume there are G different groups with individuals in each group y_{gi}, $i = 1,2,\ldots,n_g$, $n = n_1 + \cdots + n_G$ and $g = 1,\ldots,G$. Let $y_{gi} - \bar{y}$ is the total deviation ($\bar{y} = \sum_g^G \sum_i^{n_g} y_{gi}/n$), $\bar{y}_g - \bar{y}$ is the deviation of grouped mean ($\bar{y}_g = \sum_{i=1}^{n_g} y_{gi}/n_g$) around total mean, and $y_{gi} - \bar{y}_g$ is the deviation of individuals around the grouped mean. By squaring and taking the summation over both g and i then

$$\begin{aligned} \sum_{g=1}^{G}\sum_{i=1}^{n_g}(y_{gi} - \bar{y})^2 &= \sum_{g=1}^{G}\sum_{i=1}^{n_g}(\bar{y}_g - \bar{y})^2 \\ &\quad + \sum_{g=1}^{G}\sum_{i=1}^{n_g}(y_{gi} - \bar{y}_g)^2 \end{aligned}$$

That is known as the partitions the total sum of squares, in other words, the total sum of squares (TSS) is equal to the between sum of squares (BSS) plus the within sum of square (WSS), therefore

$$TSS = BSS + WSS$$

By using the absolute value instead of the square it is known that

$$|y_{gi} - \bar{y}| \le |\bar{y}_g - \bar{y}| + |y_{gi} - \bar{y}_g|$$

By taking the summation over both g and i then

$$\sum_{g=1}^{G}\sum_{i=1}^{n_g}|y_{gi} - \bar{y}| \le \sum_{g=1}^{G}\sum_{i=1}^{n_g}|\bar{y}_g - \bar{y}| + \sum_{g=1}^{G}\sum_{i=1}^{n_g}|y_{gi} - \bar{y}_g|$$

The total sum of absolute ($TSA = \sum_{g=1}^{G}\sum_{i=1}^{n_g}|x_{gi} - \bar{x}|$) is less than or equal to between sum of absolute ($BSA = \sum_{g=1}^{G}\sum_{i=1}^{n_g}|\bar{x}_g - \bar{x}|$) plus within sum of absolute ($WSA = \sum_{g=1}^{G}\sum_{i=1}^{n_g}|x_{gi} - \bar{x}_g|$), therefore,

$$TSA \leq BSA + WSA$$

To obtain equality this can be re-written as

$$TSA = BSA + WSA + R$$

where $R \leq 0$ and $R = TSA - BSA - WSA$ is the residuals.

These residuals could be separated to pure within-groups residuals and pure between-groups residuals and added to each term to obtain exact MAD partitions

$$TSA = (BSA + R_b) + (WSA + R_w) = EBSA + EWSA$$

This is shown in the following theorem.

Theorem 1

The MAD partitions about mean into the exact between sum of absolute and the exact within sum of absolute are

$$\sum_{g=1}^{G}\sum_{i=1}^{n_g}|y_{gi} - \bar{y}| = \sum_{g=1}^{G}\sum_{i=1}^{n_g}|\bar{y}_g - \bar{y}|_\delta + \sum_{g=1}^{G}\sum_{i=1}^{n_g}|y_{gi} - \bar{y}_g|_\delta \tag{5}$$

where

$$|\bar{y}_g - \bar{y}|_\delta = \begin{cases} -|\bar{y}_g - \bar{y}| & \text{if } y_{gi} \leq \bar{y} < \bar{y}_g \text{ or } y_{gi} > \bar{y} \geq \bar{y}_g \\ |\bar{y}_g - \bar{y}| & \text{else} \end{cases}$$

and

$$|y_{gi} - \bar{y}_g|_\delta = \begin{cases} -|y_{gi} - \bar{y}_g| & \text{if } \bar{y}_g < y_{gi} \leq \bar{y} \text{ or } \bar{y}_g \geq y_{gi} > \bar{y} \\ |y_{gi} - \bar{y}_g| & \text{else} \end{cases}$$

Note that $|\bar{y}_g - \bar{y}|_\delta$ is the exact between absolute and $|y_{gi} - \bar{y}_g|_\delta$ is the exact within absolute, then for mean

$$TSA_M = EBSA_M + EWSA_M$$

Proof:

Using the covariance representation of the mean absolute deviation about mean given in (4), the partitions can be written as $TSA_M = \sum_{g=1}^{G}\sum_{i=1}^{n_g}|y_{gi} - \bar{y}| = 2Cov(y_{gi}, I_{o\bar{y}}) = 2E[(y_{gi} - \bar{y})I_{o\bar{y}}] = 2\sum_{g=1}^{G}\sum_{i=1}^{n_g} t_{gi}$

$$t_{gi} = \begin{cases} y_{gi} - \bar{y}, & y_{gi} > \bar{y} \\ 0 \quad , & \text{else} \end{cases},$$

the $BSA_M = \sum_{g=1}^{G}\sum_{i=1}^{n_g}|\bar{y}_g - \bar{y}| = 2Cov(\bar{y}_g, I_{o\bar{y}}) = 2\sum_{g=1}^{G}\sum_{i=1}^{n_g} b_{gi}$

$$b_g = \begin{cases} \bar{y}_g - \bar{y}, & \bar{y}_g > \bar{y} \\ 0 \quad , & \text{else} \end{cases}$$

and the

$$WSA_M = \sum_{g=1}^{G}\sum_{i=1}^{n_g}|y_{gi} - \bar{y}_g| = \sum_{g=1}^{G} 2Cov\left(y_{gi}, I_{\bar{y}_g}\right) = 2\sum_{g=1}^{G}\sum_{i=1}^{n_g} w_{gi}$$

$$w_{gi} = \begin{cases} y_{gi} - \bar{y}_g, & y_{gi} > \bar{y}_g \\ 0 \quad , & \text{else} \end{cases}$$

Let the residuals $R = t_{gi} - b_g - w_{gi}$, we have the following cases:

1. If $w = 0$, $b > 0$, $t = 0$, we have $R = -b_g = -(\bar{y}_g - \bar{y})$ then $y_{gi} \leq (\bar{y}_g, \bar{y})$, $\bar{y}_g > \bar{y}$, this imply $y_{gi} \leq \bar{y} < \bar{y}_g$. Similarly, if $w > 0$, $b = 0$, $t > 0$, we have $R = t_{gi} - w_{gi} = \bar{y}_g - \bar{y}$, this imply $y_{gi} > \bar{y} \geq \bar{y}_g$. These residuals must be excluded from BSA to obtain PBSA.
2. Similarly, if $w > 0$, $b = 0$, $t = 0$, we have $R = -w_{gi} = -(y_{gi} - \bar{y}_g)$ this imply $\bar{y} \geq y_{gi} > \bar{y}_g$. If $w = 0$, $b > 0$, $t > 0$, we have $R = t_{gi} - b_g = (y_{gi} - \bar{y}_g)$, this imply $\bar{y} < y_{gi} \leq \bar{y}_g$. These residuals must be excluded from WSA to obtain PWSA.
3. If $w = 0$, $b = 0$, $t = 0$, we have $R = 0$ then $y_{gi} \leq \bar{y}_g \leq \bar{y}$. If $w > 0$, $b > 0$, $t > 0$, we have $R = 0$ then $y_{gi} > \bar{y}_g > \bar{y}$.
4. Two impossible cases when $w > 0$, $b > 0$ and $t = 0$ and $w = 0$, $b = 0$ and $t > 0$.

Note that the residuals are zeros when the group mean is between the overall mean and the data.

A. *Explanation of the subtracting the error term*

In the analysis of variance the total sum of squares can be written as

$$\sum_i\sum_g\left[\left(Y_{ig} - \hat{\theta}\right) + \left(\hat{\theta} - \theta\right)\right]^2 = \sum_i\sum_g\left(Y_{ig} - \hat{\theta}\right)^2 + 2\sum_i\sum_g\left(Y_{ig} - \hat{\theta}\right)\left(\hat{\theta} - \theta\right) + \sum_i\sum_g\left(\hat{\theta} - \theta\right)^2$$

In the case of $\hat{\theta} = \bar{Y}_g$ and $\theta = \bar{Y}$ the middle term is zero and the exact partitions are obtained directly. On contrast, with respect to

$$TSA = \sum_i \sum_g \left|(Y_{ig} - \hat{\theta}) + (\hat{\theta} - \theta)\right|$$

Because of the presence of absolute function the middle term does not exist where it is included inside each term. To obtain the exact partitions it is necessary to create error term from the same type and subtract it from each term.

Illustrative example

To have an idea on how the method work. Table 1 shows MAD partitions about mean for a hypothetical data. Note that $TSA = 38$, while $BSA + WSA = 36 + 18 = 54$ that overestimate TSA by $54 - 38 = 16$. On the other hand $EBSA_M + EWSA_M = 32 + 6 = 38$ that is equal exactly to TSA. Also, TSA is equal to EBSA plus EWBSA for each value.

Table 1 TSA_M partitions into $EBSA_M$ and $ESWA_M$ for a hypothetical data

g	i	y_{gi}	TSA	BSA	WSA	$EBSA_M$	$EWSA_M$
1	1	4	2	2	0	2	0
$\bar{y}_1 = 4$	2	1	5	2	3	2	3
	3	7	1	2	3	-2	3
2	1	10	4	9	5	9	-5
$\bar{y}_2 = 15$	2	20	14	9	5	9	5
3	1	2	4	4	0	4	0
$\bar{y}_3 = 2$	2	1	5	4	1	4	1
	3	3	3	4	1	4	-1
Total		48	38	36	18	32	6
$\bar{y} = 6$							

The MAD partitions about median can be shown in the same manner as MAD partitions about mean in the following theorem.

Theorem 2

The pure partitions of the total sum of absolute about median into the between sum of absolute and the within sum of absolute are

$$\sum_{g=1}^{G}\sum_{i=1}^{n_g}|y_{gi} - \tilde{y}| = \sum_{g=1}^{G}\sum_{i=1}^{n_g}|\tilde{y}_g - \tilde{y}|_\delta + \sum_{g=1}^{G}\sum_{i=1}^{n_g}|y_{gi} - \tilde{y}_g|_\delta \tag{6}$$

where

$$|\tilde{y}_g - \tilde{y}|_\delta = \begin{cases} -|\tilde{y}_g - \tilde{y}| & \text{if } y_{gi} \le \tilde{y} < \tilde{y}_g \text{ or } y_{gi} > \tilde{y} \ge \tilde{y}_g \\ |\tilde{y}_g - \tilde{y}| & \text{else} \end{cases}$$

and

$$|y_{gi} - \tilde{y}_g|_\delta = \begin{cases} -|y_{gi} - \tilde{y}_g| & if\ \tilde{y}_g < y_{gi} \le \tilde{y} \text{ or } \tilde{y}_g \ge y_{gi} > \tilde{y} \\ |y_{gi} - \tilde{y}_g| & \text{else} \end{cases}$$

Note that $|\tilde{y}_g - \tilde{y}|_\delta$ is the exact between absolute deviation about median and $|y_{gi} - \tilde{y}_g|_\delta$ is the exact within absolute deviation about median, i.e.

$$TSA_{med} = EBSA_{med} + PWSA_{med}$$

Proof: similarly as given in the mean.

4. Applications

The one way analysis of mean absolute deviation (ANOMAD) is introduced and used to test for equal population means and medians under the following assumptions.

1. The observations are random and independent samples from the populations.
2. The distributions of the populations from which the samples are selected are Laplace distribution.
3. The MADs of the distributions in the populations are equal.

It is difficult to obtain the exact sampling distributions for EBSA and EWSA, therefore, a general distribution is chosen to fit these sampling distributions. One of the families that connected to Laplace distribution is the variance-gamma distribution or also known as generalized Laplace distribution; see, [10]. More recently, the variance gamma model became popular among some financial modellers, due to its simplicity, flexibility, and an excellent fit to empirical data; [12], [13] and [11]. The variance-gamma distribution will be used to fit the sampling distributions of EBSA and EWSA via method of moments where the patterns between the moments of the data and the distribution will be equated.

A. *Fitting sampling distributions*

The random variable Y is said to have Variance-Gamma (VG) with parameters $c, \theta \in R, \nu, \sigma > 0$, if it has probability density function given by

$$f(y; c, \sigma, \theta, \nu) = \frac{2e^{\frac{\theta(y-c)}{\sigma^2}}}{\sigma\sqrt{2\pi}\nu^{\frac{1}{\nu}}\Gamma\left(\frac{1}{\nu}\right)}\left[\frac{|y-c|}{\sqrt{\frac{2\sigma^2}{\nu}+\theta^2}}\right]^{\frac{1}{\nu}-1} K_{\frac{1}{\nu}-\frac{1}{2}}\left[\frac{|y-c|\sqrt{\frac{2\sigma^2}{\nu}+\theta^2}}{\sigma^2}\right],$$

$y \in R$

Where $K_\nu(x)$ is a modified Bessel function of the third kind; see, for example, [17] and [5].

Note that there are other versions of this distribution available but this version is chosen because there is a software package in R called *gamma-variance* based on this version that be used to obtain all the simulations and graphs. The moments of this distribution are

$$E(Y) = c + \theta,$$
$$V(Y) = \sigma^2 + \nu\theta^2,$$
$$sk = \frac{2\theta^3\nu^2 + 3\sigma^2\theta\nu}{\sqrt{(\theta^2\nu + \sigma^2)^3}},$$

and

$$ku = 3 + \frac{3\sigma^4\nu + 12\sigma^2\theta^2\nu^2 + 6\theta^4\nu^3}{(\theta^2\nu + \sigma^2)^2}$$

This distribution is defined over the real line and has many distributions as special cases or limiting distributions such as Gamma distribution in the limit $\sigma \downarrow 0$ and $c = 0$, Laplace distribution as $\theta = 0$ and $\upsilon = 2$ and normal distribution as $\theta = 0, \nu = 1/r$ and $r \to \infty$. Note that if $a > 0$ then

$$aY \sim VG(ac, a\sigma, a\theta, \nu)$$

The Gamma distribution for the random variable Y is defined as

$$f(y; k, \omega) = \frac{1}{\Gamma(k)\omega^k} y^{k-1} e^{-\frac{y}{\omega}}, \quad y > 0, k, \omega > 0$$

where k and ω are the shape and scale parameters and the moments are

$$E(Y) = k\omega, \quad V(Y) = k\omega^2, \quad sk = \frac{2}{\sqrt{k}} \quad \text{and}$$
$$ku = 3 + \frac{6}{k}$$

1) MAD about median

The scaled exact between sums of absolute can be written as

$$U_1 = \frac{EBSA_{med}}{\Delta} = \frac{\sum_{g=1}^{G}\sum_{i=1}^{n_g}\left|\tilde{y}_g - \tilde{y}\right|_\delta}{\Delta}$$

Since $EBSA_{med}$ depends on one parameter G, the moments of $EBSA_{med}$ is used to fit the sampling distribution of U_1 based on VG distribution. For this purpose a simulation study is conducted to obtain the first four moments for U_1 based on simulated data from Laplace distribution (μ, Δ) with different values of n and G. Table 2 gives the simulated first four moments of U_1.

From Table 2 it is noted that there is a pattern between the mean and the variance for different n and G values where the mean is approximately $G - 1$ and the variance is twice the mean $(2(G-1))$ whatever the values of n. Therefore VG distribution is used to fit the sampling distribution of U_1 as

$$U_1 = \frac{EBSA_{med}}{\beta} \approx \text{VG}\left(c = 0, \sigma \downarrow 0, \theta = (G-1), \nu = \frac{2}{(G-1)}\right) \cong \Gamma\left(\frac{G-1}{2}, 2\right)$$

The moments can be computed as

$$E(U_1) = (G-1), \qquad V(U_1) = 2(G-1),$$
$$sk = \frac{4}{\sqrt{2(G-1)}} \quad \text{and} \quad ku = 3 + \frac{12}{G-1}$$

Hence,

$$T_1 = \frac{EBSA_{med}}{(G-1)\beta} \approx \text{VG}\left(c = 0, \sigma \downarrow 0, \theta = 1, \nu = \frac{2}{(G-1)}\right) \cong \Gamma\left(\frac{G-1}{2}, \frac{2}{G-1}\right)$$

The moments are

$$E(T_1) = 1, \qquad V(T_1) = \frac{2}{(G-1)},$$
$$sk = \frac{4}{\sqrt{2(G-1)}} \quad \text{and} \quad ku = 3 + \frac{12}{G-1}$$

Table 2 simulated mean, variance, skewness and kurtosis for $U_1$and U_2 with different values of G and n from Laplace distribution(μ, Δ) and the number of replications is 10000.

		Simulated moments for U_1				Simulated moments for U_2			
n	G	$mean$	var	sk	ku	$mean$	var	sk	ku
30	3	2.056	4.017	1.870	8.169	27.383	29.710	0.397	3.279
50	5	4.075	7.975	1.410	6.389	45.249	49.622	0.270	3.185
80	8	7.019	14.07	1.195	6.110	72.372	80.508	0.253	3.134
100	10	9.045	17.73	0.903	4.086	90.325	100.186	0.239	3.082
150	15	14.002	27.85	0.740	3.885	135.530	146.248	0.124	3.045
45	3	2.059	4.042	1.961	8.451	42.325	44.427	0.289	3.176
75	5	4.072	8.111	1.415	6.088	70.292	74.770	0.274	3.114
120	8	7.053	13.96	1.072	4.671	112.391	120.144	0.171	3.083
150	10	9.016	17.88	0.950	4.424	140.246	150.247	0.127	3.075
225	15	14.03	28.01	0.762	3.748	210.147	225.218	0.108	3.056
75	3	2.03	3.997	1.982	8.965	72.399	75.067	0.241	3.095
125	5	4.08	8.250	1.478	6.431	120.384	125.345	0.173	3.058
200	8	7.05	14.130	1.038	4.582	192.430	201.591	0.090	3.048
250	10	9.02	17.796	0.925	4.264	240.365	250.939	0.155	3.037
375	15	14.05	27.973	0.784	3.888	360.229	384.630	0.118	3.021

Moreover, Figure 1 shows the histogram of U_1 based on simulated data from Laplace distribution with fitting VG superimposed. The VG clearly gives a very good fit to U_1 for different values of G and n.

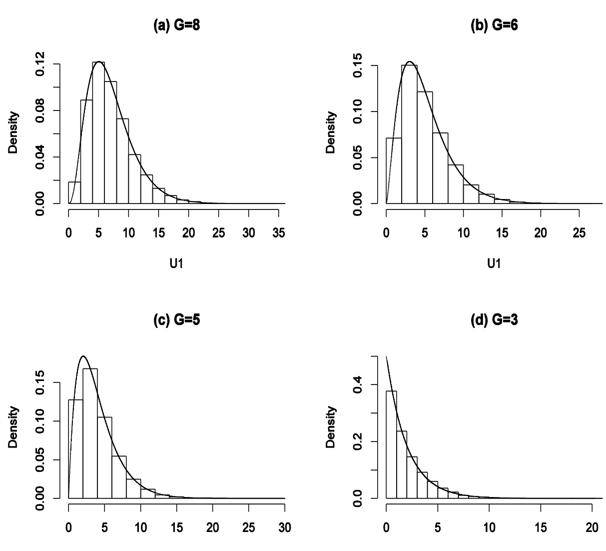

Figure 1 histogram of U_1 based on simulated data from Laplace distribution with VG distribution superimposed and (a) $G = 8$ and $n = 160$(b) $G = 6$ and $n = 120$ (c) $G = 5$ and $n = 100$ and (d) $G = 3$ and $n = 60$.

The exact within sums of absolute can be written as

$$U_2 = \frac{EWSA_{med}}{\Delta} = \frac{\sum_{g=1}^{G}\sum_{i=1}^{n_g}\left|y_{gi} - \tilde{y}_g\right|_\delta}{\Delta}$$

Since $EWSA_{med}$ depends on two parameters G and n, the moments of $EBSA_{med}$ is used to fit the sampling distribution of U_2 based on VG distribution. For this purpose a simulation study is conducted to obtain the first four moments for U_2 based on simulated data from Laplace distribution (μ, Δ) with different values of n and G. Table 2 gives the simulated first four moments of U_2 and it is noted that as expected there is a pattern between the mean and the variance for different n and G values where the mean is approximately $n - G$ and the variance is twice the mean plus G. Consequently VG distribution is used to fit the sampling distribution of U_2 as

$$U_2 = \frac{EWSA_{med}}{\beta} \approx \text{VG}\left(c = 0, \sigma = \sqrt{G}, \theta = (n - G), \nu = \frac{1}{(n - G)}\right)$$

with moments

$$E(U_2) = n - G,\ \ V(U_2) = n,\ \ sk = \frac{2n + G}{\sqrt{n^3}} \text{ and } ku = 3 + \frac{3g^2 + 6g + 6n}{n^2}$$

Therefore,

$$T_2 = \frac{PWSA_{med}}{(n-G)\beta} \approx \text{VG}\left(c = 0, \sigma = \frac{\sqrt{G}}{n-G}, \theta = 1, \nu = \frac{1}{(n-G)}\right)$$

The moments are

$$E(T_2) = 1,\ \ V(T_2) = \frac{n}{(n-G)^2},\ \ sk = \frac{2n+G}{\sqrt{n^3}} \text{ and } ku = 3 + \frac{3G^2+6G+6n}{n^2}$$

Moreover, Figure 2 shows the histogram of U_2 based on simulated data from Laplace distribution with fitting VG superimposed. The VG clearly gives a very good fit to U_2 for different values of G and n.

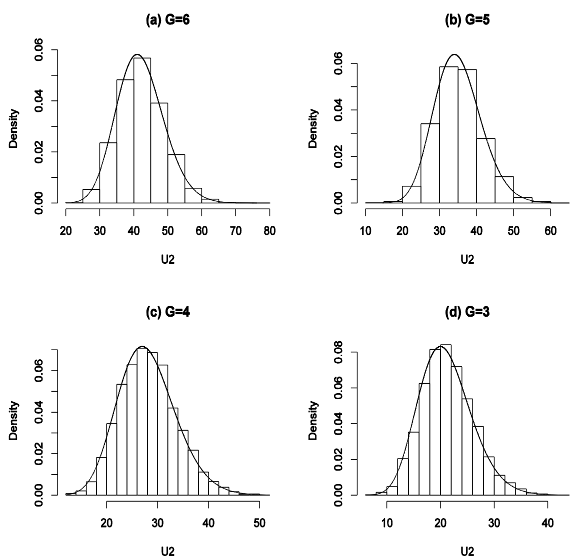

Figure 2 histogram of U_2 based on simulated data from Laplace distribution with VG distribution superimposed (a) $G = 6, n = 48$, (b) $G = 5, n = 40$, (c) $G = 4, n = 32$, and (d) $G = 3, n = 24$.

The ratio of mean EBSA to mean EWSA can be expressed as

$$R_{med} = \frac{T_1}{T_2} = \frac{\frac{EBSA_{med}}{(G-1)}}{\frac{EWSA_{med}}{(n-G)}} = \frac{(n-G)EBSA}{(G-1)EWSA} = \frac{MEBSA_{med}}{MEWSA_{med}}$$

An approximation is obtained based on gamma-variance distribution by using

$$E(R_{med}) \approx \frac{E(T_1)}{E(T_2)}$$

and

$$V(R_{med}) \approx \frac{V(T_2)E^2(T_1)}{E^4(T_2)} + \frac{V(T_1)}{E^2(T_2)} = = \frac{n}{(n-G)^2} + \frac{2}{(G-1)}$$

Therefore,

$$R_{med} \approx \text{VG}\left(c = 0, \sigma = \frac{\sqrt{n}}{n-G}, \theta = 1, \nu = \frac{2}{(G-1)}\right)$$

2) MAD about mean

The scaled EBSA deviation about mean can be written as

$$U_3 = \frac{EBSA_M}{\Delta} = \frac{\sum_{g=1}^{G}\sum_{i=1}^{n_g}\left|\bar{y}_g - \bar{y}\right|_\delta}{\Delta}$$

Since $EBSA_M$ depends on one parameter G, the moments of $EBSA_M$ is a good choice to be used to fit the sampling distribution of U_3 based on VG distribution. For this purpose a simulation study is conducted to obtain the first four moments for U_2 using simulated data from Laplace distribution (μ, Δ) with different values of n and G. Table 4 gives the simulated first four moments of U_3. From Table 3 it is noted that there is a pattern between the mean and the variance for different n and G values where the mean is approximately $G-1$ and the variance is three times the mean $(3(G-1))$ whatever the value of n. Therefore VG distribution is used to fit the sampling distribution of U_3 as

$$U_3 = \frac{EBSA_M}{\Delta} \approx \text{VG}\left(c = 0, \sigma = \sqrt{(g-1)}, \theta = (G-1), \nu = \frac{2}{(G-1)}\right)$$

The moments can be computed as

$$E(U_3) = (G-1),\quad V(U_3) = 3(G-1),\quad sk = \frac{14}{\sqrt{27(G-1)}} \text{ and } ku = 3 + \frac{102}{9(G-1)}$$

Hence,

$$T_3 = \frac{EBSA_M}{(G-1)\Delta} \approx \text{VG}\left(c = 0, \sigma = \frac{1}{\sqrt{G-1}}, \theta = 1, \nu = \frac{2}{(G-1)}\right)$$

Table 3 simulated mean, variance, skewness and kurtosis for U_3 and U_4 with different values of G and n from Laplace distribution(μ, Δ) and number of replications is 10000

		Simulated moments for U_3				Simulated moments for U_4			
n	G	$mean$	var	sk	ku	$mean$	var	sk	ku
30	3	2.079	5.980	1.795	7.953	27.871	31.752	0.366	3.199
50	5	4.052	11.606	1.302	5.534	46.033	55.172	0.284	3.118
80	8	7.043	20.706	0.976	4.339	72.884	86.579..	0.215	3.084
100	10	9.032	26.759	0.867	4.271	91.055	110.502	0.199	3.061
150	15	14.015	41.891	0.734	3.815	136.041	163.125	0.161	3.047
45	3	2.063	5.972	1.781	7.901	42.875	48.324	0.299	3.130
75	5	4.053	11.752	1.293	5.431	70.771	80.412	0.245	3.071
120	8	7.021	20.811	0.952	4.353	112.895	128.012	0.182	3.049
150	10	9.019	26.522	0.871	4.301	141.049	160.125	0.164	3.038
225	15	14.025	41.902	0.733	3.791	211.052	238.752	0.133	3.018
75	3	2.033	5.941	1.801	7.899	73.066	77.432	0.244	3.076
125	5	4.031	11.842	1.311	5.432	120.795	128.840	0.178	3.054
200	8	7.019	20.811	0.962	4.360	192.82	208.176	0.147	3.046
250	10	9.022	26.522	0.886	4.331	241.034	258.552	0.126	3.037
375	15	14.017	41.902	0.724	3.801	360.974	390.05	0.101	3.022

Moreover, Figure 4 shows the histogram of U_3 based on simulated data from Laplace distribution with fitting VG superimposed. The VG clearly gives a very good fit to U_3 for different values of G and n.

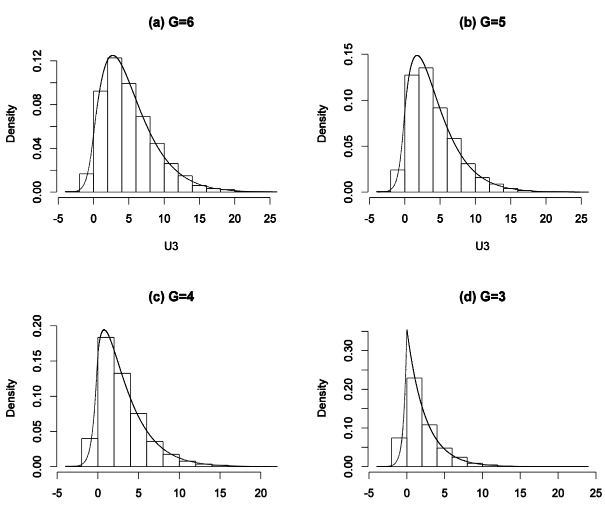

Figure 4 histogram of U_3 based on simulated data from Laplace distribution with VG distribution superimposed and (a) $G = 6$ and $n = 60$(b) $G = 5$ and $n = 50$ (c) $G = 4$ and $n = 40$ and (d) $G = 3$ and $n = 30$.

The EWSA deviation about mean can be written as

$$U_4 = \frac{EWSA_M}{\Delta} = \frac{\sum_{g=1}^{G} \sum_{i=1}^{n_g} \left| y_{gi} - \bar{y}_g \right|_\delta}{\Delta}$$

Since $EWSA_M$ depends on two parameters G and n, the moments of $EWSA_M$ could be used to fit the sampling distribution of U_4 based on VG distribution. For this purpose a simulation study is conducted to obtain the first four moments for U_4 based on simulated data from Laplace distribution with different values of n and G. Table 3 gives the simulated first four moments of U_4. From Table 3 it is noted that there is a pattern between the mean and the variance for different n and G values where the mean is approximately $n - G + 1$ and the variance is equal to the mean plus $2(G-1)$. Therefore VG distribution is used to fit the sampling distribution of U_4 as

$$U_4 = \frac{PWSA_M}{\beta} \approx \text{VG}\left(c = 0, \sigma = \sqrt{2(G-1)}, \theta = (n - G + 1), \nu = \frac{1}{(n - G + 1)}\right)$$

with moments

$$E(U_4) = n - G + 1,\ \ V(U_4) = n + G - 1,\ \ sk = \frac{2n + 4G - 4}{\sqrt{(n + G - 1)^3}}$$

and

$$ku = 3 + \frac{6n^2 - 6g^2 + 12gn + 12g - 12n + 42}{(n - g + 1)(n + g - 1)^2}$$

Therefore,

$$T_4 = \frac{PWSA_M}{(n - G + 1)\beta} \approx \text{VG}\left(c = 0, \sigma = \frac{\sqrt{2(G-1)}}{(n - G + 1)}, \theta = 1, \nu = \frac{1}{(n - G + 1)}\right)$$

The moments are

$$E(T_4) = 1,\ \ V(T_4) = \frac{n + G - 1}{(n - G + 1)^2},\ \ sk = \frac{2n + 4G - 4}{\sqrt{(n + G - 1)^3}}$$

and

$$ku = 3 + \frac{6n^2 - 6g^2 + 12gn + 12g - 12n + 42}{(n - g + 1)(n + g - 1)^2}$$

Figure 4 shows the histogram of U_4 based on simulated data from Laplace distribution with fitting VG superimposed. The VG clearly gives a very good fit to U_4 for different values of G and n. Note that more simulation results for different G and n are available from the author upon request.

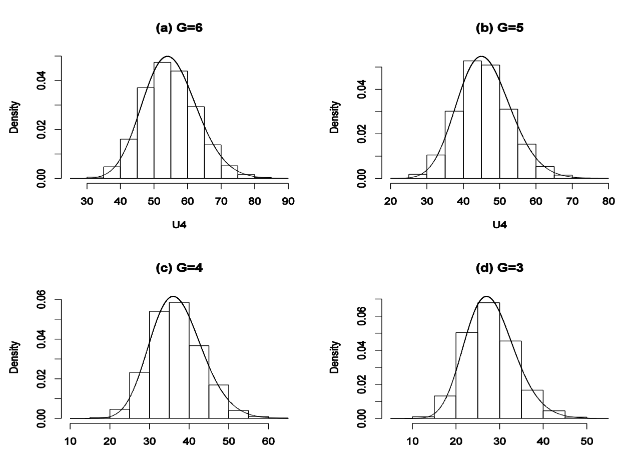

Figure 5 histogram of U_4 based on simulated data from Laplace distribution with VG distribution superimposed and (a) $G = 6$ and $n = 60$(b) $G = 5$ and $n = 50$ (c) $G = 4$ and $n = 40$ and (d) $G = 3$ and $n = 30$.

The ratio of mean between absolute deviations to mean within absolute deviation can be expressed as

$$R_M = \frac{T_3}{T_4} = \frac{\dfrac{PBSA_M}{(G-1)}}{\dfrac{PWSA_M}{(n - G + 1)}} = \frac{(n - G + 1)PBSA}{(G - 1)PWSA} = \frac{MBSA_M}{MWSA_M}$$

An approximation to R_M is obtained based on gamma-variance distribution by using

$$E(R_M) \approx \frac{E(T_3)}{E(T_4)} = 1$$

and

$$V(R_M) \approx \frac{V(T_4)E^2(T_3)}{E^4(T_4)} + \frac{V(T_3)}{E^2(T_4)} = \frac{n + G - 1}{(n - G + 1)^2} + \frac{3}{(G - 1)}$$

Hence, the approximation gamma-variance is

$$R_M \approx \text{VG}\left(c = 0, \sigma = \sqrt{\frac{n + G - 1}{(n - G + 1)^2} + \frac{1}{G - 1}}, \theta = 1, \nu = \frac{2}{(g - 1)}\right)$$

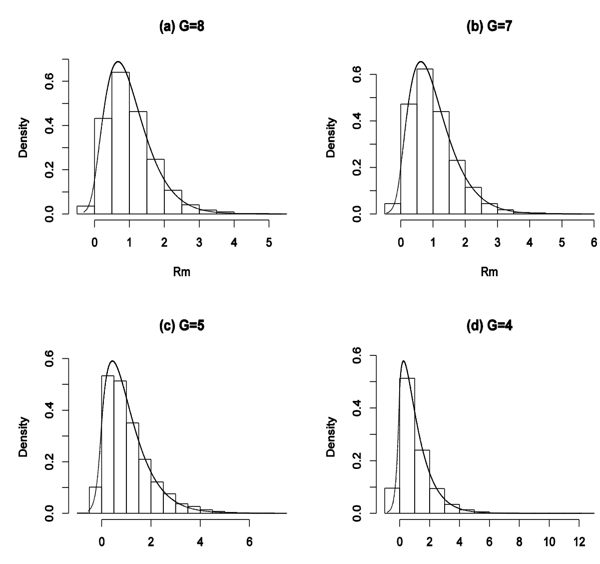

Figure 6 histogram of R_{med} based on simulated data from Laplace distribution with VG distribution superimposed (a) $G = 8, n = 160$, (b) $G = 7$ and $n = 140$, (c) $G = 5$ and $n = 100$ and (d) $G = 3, n = 60$.

Figure 6 shows the histogram of an approximate distribution of R_m. It is clearly that it gives a very good fit to R_m.

B. Tests for equal medians and means

1) Test for equal medians

The null hypothesis H_0tested in one way ANOMAD is that the population medians from which the G samples are selected are equal

$$H_0: \nu_1 = \nu_2 = \nu_3 = \cdots = \nu_G$$

The alternatively hypothesis H_a is that at least two of the group medians are significantly different. Table 4 gives a summary of ANOMAD for medians.

Table 4 summary ANOMAD for medians

variation	Sum of absolute	Divisor	MAD estimate (mean absolute)	R_{med}
Between	$EBSA_{med}$	$G-1$	$MEBSA_{med} = \frac{EBSA_{med}}{G-1}$	$\frac{MEBSA_{med}}{MEWSA_{med}}$
Within	$EWSA_{med}$	$n-G$	$MEWSA_{med} = \frac{EWSA_{med}}{n-G}$	
Total	TSA			

To test for the assumption of Laplace distribution, the function *laplace.test()* in package *lawstat in R-software* is used where it gives five goodness of fit for the Laplace distribution based on the work of [16]. A one way ANOMAD is used to examine if students' scores on a standardized test is a function of the teaching method. The independent variable represented the three different types of teaching method: 1) lecture only; 2) hands-on only; and 3) both. The dependent variable is the students' score on a standardized test. See Table 5 for the sample data with means, medians and MAD reported for each of the three groups. To test the assumption of Laplace distribution, Kolmogorov-Smirnov (D) is used from package *lawstat* in R-software. The results for the three groups are given in Table 5 where p-values more than 0.01, 0.05 and 0.10, therefore, the assumption of Laplace cannot be rejected. Because the maximum MAD to minimum MAD is 1.14, the assumption of homogeneity of MAD's cannot be rejected.

Table 5 students' scores for three different methods of teaching and Laplace goodness of fit using Kolmogorov-Smirnov test

Teaching methods			Laplace test			
L.	H.on	B	Kolmogrrov-simrnov			
65	82	87	Group	p-value		
63	70	86	Lect.	0.67		
78	70	78	Lab.	0.68		
88	75	88	Both	0.73		
59	37	72				
71	70	98		Lecture	Hands-on	Both
70	74	95	Mean	70	68	84
72	63	85	Median	71	70	86
70	81	72	MAD_M	5.7	6.12	6.5
81	70	90	MAD_{med}	5.7	5.9	6.4
72	68	75				
68	69	84				
56	63	86				
65	64	92				
71	68	74				

Table 6 ANOMAD of testing equal medians for standardized test scores

variation	Sum of absolute	Divisor	MAD estimate (mean absolute)	R_{med}	$qVG^*_{0.95}$
Between	202	2	101	24.82	3.02
Within	171	42	4.07		
Total	373				

*This value from Variance-Gamma package in R-software

where $R_{med} = 24.82 > Critical = qVG_{0.95} = 3.02$, H_o is rejected, i.e., this indicates that not all three groups of the teaching methods resulted in the same standardized test score.

2) Test for equal means

The null hypothesis H_0 tested in one way ANOMAD is that the population means from which the G samples are selected are equal

$$H_0: \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_G$$

The alternatively hypothesis H_a is that at least two of the group means are significantly different.
Table 4 gives summary of ANOMAD for means.

Table 7 summary ANOMAD for means

Variation	Sum of absolute	Divisor	MAD estimate (mean absolute)	R_M
Between	$EBSA_M$	$G-1$	$MEBSA_M = \frac{EBSA_M}{G-1}$	$\frac{MEBSA_M}{MEWSA_M}$
Within	$EWSA_M$	$n-G+1$	$MEWSA_M = \frac{EWSA_M}{n-G+1}$	
Total	TSA			

Table 8 gives ANOMAD to test equal means for standardized test scores.

Table 8 ANOMAD of testing equal means for standardized test scores

variation	Sum of absolute	Divisor	MAD estimate (mean absolute)	R_M	$qVG^*_{0.95}$
Between	197	2	98.5	22.6	3.45
Within	187	43	4.35		
Total	384				

*This value from Variance-Gamma package in R-software

where $R_{med} = 22.6 > Critical = qVG_{0.95} = 3.45$, H_o is rejected, i.e. this indicates that not all three groups of the teaching methods resulted in the same standardized test score.

C. *Effect size*

Effect size (ES) is a measure of practical significance where it is defined as the degree to which a phenomenon exists where any observed difference between, for example, two sample means can be found to be statistically significant when the sample sizes are sufficiently large. In such a case, a small difference with little practical importance can be statistically significant. On the other hand, a large difference with apparent practical importance can be non-significant when the sample sizes are small. Therefore, ES provide another measure of the magnitude of the difference expressed in standard deviation units in the original measurement. Thus, with the test of statistical significance (e.g., the *F* statistic) and the interpretation of the effect size (*ES*), the researcher can address issues of both statistical significance and practical importance. Standardized ES measures typically employed in behavioural and social sciences research; see, [1] and [2].

The first type of standardized ES measure is

$$\eta^2 = \frac{S_B^2}{S_B^2 + S_W^2}$$

Where S_B^2 and S_W^2 are between group variance and within group variance; see, for example, [3]. Another measure of the strength of the association between the independent variable and the dependent variable in ANOVA is ω^2 that indicates the proportion of the total variance in the dependent variable that is accounted for by the levels of the independent variable. This is analogous to the coefficients of determination r^2. The formula for ω^2 is

$$\omega^2 = \frac{SS_B - (G-1)MS_W}{SS_T + MS_W}$$

See, for example Cohen (1988).
These two measures are defined in terms of ANOMAD as

$$\eta_{mad} = \frac{EBSA}{EBSA + EWSA}$$

and

$$\omega_{mad} = \frac{EBSA - (G-1)MEWSA}{TSA + MEWSA}$$

Therefore, η_{mad} by using median is $202/373 = 0.54$ and mean is $197/384 = 0.51$ while ω_{mad} by using median is $(202 - 2(4))/(373 + 4) = 0.51$ and mean is $(196 - 2(4.4))/(384 + 4.4) = 0.48$. For example, when ω_{mad} is 0.51 this means that the independent variable in the ANOMAD accounts for 51% of the total variance in the dependent variable.

5. CONCLUSION

There are many areas in sciences where the normal distribution is not an appropriate approximation and the Laplace distribution provides a good approximation to the data. In these cases, when the tests of equal means and medians are needed, the ANOMAD will be appropriate. The ANOMAD was derived using the idea of creating error terms from the same type (BSA and WSA) and subtract it from each term to obtain the exact partitions.

The ANOMAD had important information about the shifts in means and medians of the groups and that studied by fitting variance-gamma distribution to EBSA and EWSA and test for equal means or medians. Because of no square in MAD, the ANOMAD ensured high stability of statistical inference. Finally, the effect sizes were re-expressed and studied in terms of ANOMAD.

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