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# Linear Moments: An Overview

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**Abstract:** Many statistical techniques are based on the use of linear combinations of order statistics that called linear moments. L-moments are a sequence of statistics used to summarize the shape of a probability distribution. They are linear combinations of order statistics analogous to conventional moments, and can be used to calculate quantities analogous to standard deviation, skewness and kurtosis, termed the L-scale, L-skewness and L-kurtosis respectively. In this paper an overview for recent works in L-moments is presented.

**Keywords:** Order statistics, L-moments, TL-moments, LQ-moments, Quantile function.

## 1. Introduction

Many statistical techniques are based on the use of linear combinations of order statistics (or quantile function), but there has not been developed a unified theory of estimation covering the characterization of probability distribution, until Hosking (1990) introduced L-moments, as an alternative to the classical moments. So, L-moments are summary statistics for probability distributions and data samples. They provide measures of location, dispersion, skewness, kurtosis, and other aspects of the shape of probability distributions or data samples. However, The L in L-moments emphasizes the construction of L-moments from linear combinations of order statistics.

Sillitto (1969) has derived an approximation to the inverse distribution function (the quantile function) in terms of population L-moments without referring to L-moments. Also, he gave the sample version of the quantile function in terms of sample L-moments without studying its properties. A formal and comprehensive treatment of L-moments was developed by Hosking (1990), who established foundational results supporting a new methodology in data analysis and inference based on L-moments.

Hosking (1990) concluded that the L-moments, as a function of the quantile function, have various theoretical advantages over the classical moments. For example, for L-moments of a probability distribution to be meaningful, we require only that the distribution has a finite mean; no higher-order moments need be finite. Similarly, in order to the standard errors of L-moments to be finite, only the distribution is required to have finite variance and no higher-order moments need be finite. Also, though moment ratios can be arbitrarily large, sample moment ratios have algebraic bounds but sample L-moment ratios can take any values that the corresponding population quantities can.

In addition, L-moments have properties that hold in a wide range of practical situations. L-moments also give asymptotic approximations to sampling distributions better than classical moments and provide better identification of the parent distribution which generated a particular data sample (see Hosking (1990)). Furthermore, L-moments are less sensitive to outlying data values and yield, sometimes, more efficient parameter estimates than the maximum likelihood estimates (Vogel and Fennessey (1993)).

But, the main shortage in L-moments is that they do not exist for the distribution which has infinite mean and less efficient for heavy tail distributions.

Mudholkar and Hutson (1998), introduced, as a result of the previous shortage, the LQ-moments, as a robust version of L-moments in which expectation of the conceptual order statistics (in the definition of population L-moments) is replaced by a class of robust location measures defined in terms of simple linear combinations of symmetric quantiles of the distributions of the order statistics, such as median and trimean.

Elamir and Seheult (2003) introduced Trimmed L-moments (TL-moments) as an alternative to LQ-moments and natural generalization of L-moments that do not

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require the mean of underlying distribution to exist. TL-moments depend on giving zero weight to extreme observations. Therefore, they are defined for heavy tailed distributions where they do not involve some values at the extreme ends of the distribution.

As interested in statistical modeling using heavy-tailed distributions is increasing, so is the important of(a) the potential by the L-moment and TL-moment approaches. The need of linear moments has been developed in support of regional frequency analysis in environmental science, which treats the quantile of distributions of variables such as annual maximum precipitation, stream flow, wind speed observed at each site in a given network. Hosking and Wallis (1997) provide an excellent exposition. Also linear moments approach has special utility in applications where descriptive estimates (location, spread, skewness, and kurtosis) more stable than the usual central moments are critically needed. Such concern arise, for example, in volatility estimation in financial risk management involving market variables such as stock indices, interest rates; see, Serfling (1980) , Embrechts et al. (1997), Leonowicz et al. (2005) and Willinger et al. (1998).

TL-moments have various theoretical advantages. For example, TL-moments give more robust estimators than L-moments in the presence of outliers. Moreover, population TL-moments may be well defined where the corresponding population L-moments (or central moments ) do not exist , for example, the first population TL-moment is well defined for a Cauchy distribution, but the first population L-moment, the population mean, does not exist (see, Elamir and Seheult (2003)).

Also, their sample variance and covariance can be obtained in closed form (see, Elamir and Seheult (2003)). In addition, TL-moments ratios are bounded for any trimmed value (see, Hosking (2007 and Elamir et al (2010)).

Hosking (2007) has derived of approximation to the quantile function in terms of population TL-moments. Elamir (2009) introduced properties of this approximate function by minimizing the weighted mean square error between the population quantile function and its TL-moments representation. Also, he studied properties of the corresponding sample estimator. He concluded that the estimators have a good approximation to population quantile for a broad class of probability distribution functions.

Also, Elamir (2010) derived an optimal choice for the amount of trimming from known distributions, based on the minimum sum of the absolute value of the errors between the quantile probability function and its TL-moments representation. Nair and Vineshkumar (2010) study the properties of L-moments of residual life in the context of modeling lifetime data. They introduced characterizing life distributions and other applications. The role of certain quantile functions and quantile-based concepts in reliability analysis are also investigated.

Several studies had been done using and applying the method of TL-moments as natural generalization of L-moments:

Some of these studies concerned to introduce theoretical results for TL-moments method are as follows:

Karvanen (2006) proposed parametric families from quantile functions of distributions. This class of parametric families is called quantile mixtures, which contain a wide range of different distributions that have practical importance. He introduced two parametric families: the normal-polynomial quantile mixture and the Cauchy-polynomial quantile mixture. And he used the methods of L-moments and TL-moments to estimate the parameters of the two quantile mixtures respectively. The proposed quantile mixtures are applied to model monthly, weekly and daily returns of some major stock indexes.

Abdul-Moniem (2007) applied the method of L-moment and TL-moment estimators to estimate the parameters of exponential distribution.

Asquith (2007) derived analytical solutions for the first five L-moments and TL-moments (in case of symmetrical trimming (1,1) in terms of a four-parameters generalized Lambda distribution (GLD) in asymmetric case. And, because that asymmetric GLD needs numerical methods to compute the parameters from the L-moments or TL-moments, algorithms are suggested for parameter estimation. Application of the GLD using both L-moments and TL-moment parameter estimates from example data is demonstrated, a small simulation study of the 98th percentile (far-right tail) is conducted for a heavy-tail GLD with high-outlier contamination. The simulations showed, with respect to estimation of the 98th-percent quantile, that TL-moments are less biased (more robust) in the presence of high-outlier contamination.

Hosking (2007) derived some further theoretical results for TL-moments. He defined the TL-moments in terms of shifted jacobi polynomials, and he introduced a representation for the quantile function as a weighted function of orthogonal polynomials in which the coefficients are related to TL-moments.

Abdul-Moniem and Selim (2009) derived the TL-moments and L-moments of the generalized Pareto distribution (GPD), and they used it to obtain the first four TL-moments and L-moments. They introduced the TL-skewness, L-skewness, TL-kurtosis and L-kurtosis for the GPD. They used the TL-moments and L-moments to estimate the parameters for the GPD. Also, they introduced a numerical illustrate for the new results.

Elamir (2009) introduced properties of a new class of approximations to population quantile and sparsity functions based on TL-moments by minimizing the weighted mean square error between the population quantile function and its TL-moments representation. Also, he studied properties of the corresponding sample estimator of population quantile in terms of sample L-moments and Jacobi polynomials from uniform distribution, $t-$distribution and generalized Pareto distribution (GPD). He concluded that the estimators have a good approximation to population quantile for a broad class of probability distribution functions. An example is given that illustrates the benefits of the proposed method.

Elamir et al. (2009) introduced Fractional Linear moments (FL-moments) as a natural generalization of TL-moments. FL-moments take account of the data at the tail of the distribution where the sample FL-moments captures all the information in the sample about the population counterparts by assigning less weight for extreme values and provide simple and effective ways of estimating the parameters and making inferences for distributions with heavy tails in which the mean does not exist. The quantile function in terms of FL-moments as a weighted sum of Jacobi polynomials is obtained.

Also, Elamir (2010) derived an optimal choice for the amount of trimming from known distributions, based on the minimum sum of the absolute value of the errors between the quantile probability function and its TL-moments representation. Moreover, he introduced a simulation-based approach to choose an optimal amount of trimming, by computing the estimator variance for range of trimming and choose the one which has less variance. Several examples are given to show the benefits of the methods.

Maillet and Medecin (2010) introduced the relation between the $rth$ TL-moments and the first TL-moments with generalized trimming, symmetric $(t,t)$ and asymmetric $(t_1,t_2)$ trimming. Indeed, it is sufficient to compute TL-moments of order one to obtain all TL-moments. They decided the importance of this relation which helps to enable easier calculations for the $rth$ TL-moments with any trimming and L-moments as particular cases of the $rth$ TL-moments with zeros trimming. They underlined that the TL-moments approach is a general frame-work that encompasses the L-moments, LH-moments (TL-moments with $t_1=0$ ) and the LL-moments (TL-moments with $t_2=0$ ).

Abu El-Magd (2010) introduced the TL-moments and LQ-moments of the exponentiated generalized extreme value distribution (EGEV). She introduced The TL-moment estimators, L-moments estimators, LQ-moment estimators and the method of moment estimators (classical estimators) for the EGEV distribution. A numerical simulation compares these methods of estimation mainly with respect to their biases and root mean squared errors (RMSEs) will be obtained. Also, she derived the true formulae for the rth classical moments and the probability weighted moments (PWMs) for the EGEV distribution to correct the Adeyemi and Adebanji (2006) formulae for the EGEV. (b) On the other hand, TL-moments approach has been used and applied in an analysis to determine the best-fitting distributions, as analogous of L-moments which has many applications in this field (see Enayat (2009)).

Ariff (2009) used the TL-moments, in case symmetrically trimmed by one and two conceptual sample values respectively and L-moments for an analysis to determine the best fitting distribution to the data of maximum daily rainfalls measured over stations in Selangor and Kuala Lumpur. He used, for this propose, the normal (NO), logistic (LOG), extreme value (GEV) and generalized Pareto (GP) distributions. He generalized logistic (GLO), extreme value type I (EV), generalized estimated the parameters of the previous distributions by using the TL-moments and L-moments methods. He determined the most suitable distribution a cording to the mean absolute deviation index (MADI), mean square deviation index (MSDI) and correlation, r. The L-moments method showed that the generalized logistic (GLO) distribution is the best distribution whilst TL-moments method concluded that the extreme value typeI (EV) and generalized extreme value (GEV) distributions are the most suitable distributions. That results to fit the data of maximum daily rainfalls for stations in Selangor and Kuala Lumpur.

Shabri et al (2011a) derived the TL-moments with one smallest value were trimmed from the conceptual sample (TL-moments (1,0)) for the generalized logistic (GLO) distribution. They estimated the parameters of the generalized logistic (GLO) distribution by using the TL-moments with one smallest value were trimmed. They compared the performance of TL-moments with one smallest value were trimmed with L-moments and TL-moments through Monte Carlo simulation and stream flows data over station in Terengganu, Malaysia. The result showed that in certain cases, TL-moments with one smallest value were trimmed is a better option as compared to L-moments and TL-moments in modelling those series.

Deka and Borah (2012) used TL-moments method in an analysis to determine the best fitting distribution to ten stream flow gauging sites of the North Brahmaputra region of India. Three parameters extreme value distributions: generalized extreme value (GEV) distribution, generalized logistic (GLO) distribution and generalized Pareto (GP) distribution are fitted for this purpose. They valuated the performances of the three distributions using three goodness of fit tests. namely relative root mean square error, relative mean absolute error and probability plot correlation coefficient. Further, TL-moments ratios diagram is also used to confirm the goodness of fit for the above three distributions. Finally, they compared goodness of fit test results and concluded that GEV distribution is the best fitting distribution for describing the annual flood peak series for the majority of the stations in North Brahmaputra region of India when the parameters are estimated by using TL-moments method.

Shabri et al (2011b) they introduced a comprehensive evaluation of the L-moments and TL-moments methods, by first revisiting regional frequency analysis based on the L-moments by Hosking and Wallis (1993; 1997). The analysis was based on daily annual maximum rainfall data from 40 stations in Selangor Malaysia. They derived TL-moments for the generalized extreme value (GEV) distributions and generalized logistic (GLO) distributions, and they used TL-moments for the generalized Pareto distributions (GP) from Elamir and Seheult (2003). And they used the TL-moments ratios diagram and Z-test to determine the best-fit distribution. They showed, by comparison between the two approaches, that the L-moments and TL-moments produced equivalent results. While, GLO and GEV distributions identified as the most suitable distributions for representing the statistical properties of extreme rainfall in Selangor. Finally, they concluded, by Mont e Carlo simulation, that the method of TL-moments more efficient for lower quantile estimations but the L-moments method does outperform for higher quantile estimations.

Ahmad et al (2012) used TL- moments method in an analysis to determine the best-fitting distributions to represent the annual series of maximum streamflow data over 12 stations in Terengganu, Malaysia. They estimated the parameters of the generalized Pareto (GP), generalized logistic (GLO), and generalized extreme value (GEV) distributions by using the TL-moments method with different trimming value. They examined the influence of TL-moments on estimated probability distribution functions by evaluating the relative root mean square error and relative bias of quantile estimates through Monte Carlo simulations. They used the box-plot to show the location of the median and the dispersion of the data, which helps in reaching the decisive conclusions. For most of the cases, the results show that TL-moments with one smallest value were trimmed from the conceptual sample (TL-moments (1, 0)) of GP distribution was the most appropriate in majority of the stations.

## 2. Probability Weighted Moments

Let $X_1, X_2, \ldots$ be a sequence of independent random variables from a distribution with density function $f(x)$, quantile function $x(F) = F^{-1}(x) = Q(u)$ where $0 < u < 1$, cumulative distribution function $F(x) = F_X = F$, the population mean $\mu_X = E(X) = E[x(F)]$, and denote the corresponding order statistics by $X_{1:n}, \ldots, X_{n:n}$. The mean $\mu = E(X)$ and $\sigma$ is the standard deviation of the distribution.

The probability weighted moments are a generalization of the usual moments of a probability distribution (Greenwood et al., 1979). The probability weighted moments are

$$M_{p,r,s} = E[X^p\{F(X)\}^r\{1 - F(X)\}^s]$$

Where $p, r$ and $s$ are real numbers. PWM are likely to be most useful when quantile function $x(F)$ can be written in closed form, so we can rewrite

$$M_{p,r,s} = \int_0^1 [x(F)]^p \, F^r \, (1-F)^s dF$$

The quantities $M_{p,0,0}$ are the usual non-central moments. When $r$ and $s$ are integers, $F^r(1-F)^s$ may be expressed as a linear combination of either powers of $F$ or powers of $(1-F)$, so it is natural to summarize a distribution either by the moments $M_{1,r,0}$ or $M_{1,0,s}$ , where

$$\beta_r = M_{1,r,0} = E[X\{F(X)\}^r], \quad r = 0,1,2, \ldots$$

and,

$$\beta_s = M_{1,0,s} = E[X\{1 - F(X)\}^s], \quad s = 0,1,2, \ldots$$

where the expected value of order statistics is

$$E(X_{r:n}) = \frac{n!}{(r-1)!\,(n-r)!}\int_0^1 x(F)\, F^{r-1}\,(1-F)^{n-r} dF$$

Therefore the probability weighted moments can be written in terms of expected value of order statistics as

$$\beta_r = \int_0^1 x(F) F^r dF = \frac{E(X_{r+1:r+1})}{r+1}, \qquad r = 0,1, \ldots$$

and

$$\beta_s = \int_0^1 x(F)(1-F)^s dF = \frac{E(X_{1:s+1})}{s+1}, \qquad s = 0,1, \ldots$$

Note that $\beta_r$ and $\beta_s$ are used in many applications such as linear moments (see Hosking et al (1990)), estimation of the generalized extreme value distribution.

### 3. L-MOMENTS

Sillito (1969) and Hosking (1990) defined the population L-moments $\lambda_r$ as follows

$$\lambda_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r-k:r}),$$
$$r = 1,2, \dots$$

The L-moments in terms of quantile function can be written as

$$\lambda_r = \int_0^1 P_{r-1}(u) Q(u) du,$$

where

$$P_r(u) = \sum_{k=0}^{r} c_{r,k} u^k$$

and

$$c_{r,k} = (-1)^{r-k} \binom{r}{k} \binom{r+k}{k}$$

### 4. TRIMMED L-MOMENTS

Elamir and Seheult (2003) defined the trimmed L-moment (TL-moments) in terms of expected values as

$$\lambda_r^{(t_1,t_2)} = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E\left(X_{r+t_1-k:r+t_1+t_2}\right),$$
$$r = 1, \dots, t_1, t_2 = 0,1, \dots$$

where $t_1$ and $t_2$ are the amounts of trimming need to be chosen; see, Elamir (2010). Many studies have been done by researcher regarding TL-moments; see, for example, Asquith (2007), Hosking (2007), and Elamir (2010). Note that L-moments is a special case of TL-moments for $t_1 = t_2 = 0$.

### 5. LQ MOMENTS

This method due to Mudholkar and Hutson (1998) where the expected value is replaced by its median or some others population location measure as

$$\xi_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \tau_{p,a}(X_{r-k:r}), \qquad r = 1,2, \dots$$

where $\tau_{p,a}(X_{r-k:r})$ is a quick measure of the location of the sampling distribution of the order $X_{r-k:r}$.

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