



TL-Moments of Residual Life

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Abstract: although the mean, variance and coefficient of variation of residual life which are based on the usual moments of the residual life distribution are extensively used in reliability analysis. It has been established in various theoretical and empirical studies that the TL-moments have some advantages over the usual moments in many situations. Accordingly in the present paper we study the properties of L-moments of residual life in the context of modeling lifetime data, characterizing life distributions and other applications. The role of certain quantile functions and quantile-based concepts in reliability analysis are also investigated.

Keywords: linear moments, order statistics, reliability, quantile function

1. INTRODUCTION

The concept of residual life, based on the information that a unit has survived in its function for a specified period of time, has been fundamental in developing various measures and methodologies in reliability and survival analysis. Of these measures, the mean, variance and coefficient of variation of residual life based on the moments of the residual life distribution are used for modeling lifetime data, characterizing life distributions, developing criteria for ageing, ordering distributions and in evolving strategies for maintenance and repair of equipments. Recently there is considerable interest in the application of L-moments introduced by Hosking (1990) as an alternative to the conventional moments in modeling and inference problems. It has been shown theoretically and empirically (see for e.g. Hosking and Wallis, (1997); Sankarasubramanian and Sreenivasan, (1999); Perez et al., 2003; Asquith, 2007) that L-moments have several advantages over usual moments, like smaller sampling variance, robustness against outliers, easier characterization of distributional shapes. Further, finiteness of the mean guarantees the existence of all other L-moments so that a distribution whose mean exists is uniquely determined by its sequence of L-moments. This property is not always shared by the usual moments. Specification of a distribution by its L-moments even when the conventional moments do not exist is another advantage in modeling problems. These considerations apply to lifetime data analysis as well, especially in the case of heavy tailed distributions where

the asymptotic efficiency of sample moments in inference problems is not very attractive. In view of these facts, it seems worthwhile to investigate the properties of L-moments of residual life and their relevance in various aspects of reliability analysis. This problem does not appear to have been considered in the literature. But, the main shortage in L-moments is that they do not exist for the distribution which has infinite mean and less efficient for heavy tail distributions. Trimmed L-moments (TL-moments) method which introduced by Elamir and Seheult (2003) is a generalization of the L-moments method. They depend on trimming some observations from both tails of the distribution by assigning these extreme values zero weights. TL-moments have certain advantages over L-moments and conventional moments. A population TL-moment can exist when the corresponding population L-moment or conventional moment does not exist. Moreover, sample TL-moments are more resistant to outliers. TL-moments approach is a useful tool in the theoretical and applied statistics. See, for example, Karvanen (2006), Hosking (2007), Karvanen and Nuutinen (2007) Asquith (2007), Abdul-Moniem (2007), Abdul-Moniem and Selim (2009) and Elamir (2009). Nair and Vinesh Kumar (2010) studied the properties of L-moments of residual life in the context of modeling lifetime data. They introduced characterizing life distributions and other applications. The role of certain quantile functions and quantile-based concepts in reliability analysis are also investigated. In this paper we study the properties of TL-moments of residual life in the context of modeling lifetime data. Our



discussion in this direction is predominantly based on quantile functions instead of distribution functions. There exist simple forms of quantile functions that can be considered for reliability modeling for which the corresponding distribution functions do not have closed forms making the use of concepts defined in terms of distribution function in the analysis, difficult. Secondly in many cases quantile based concepts have better analytic tractability.

In section 2 we define the TL-moments of residual life and discuss its properties. Some families of quantile functions which have potential to be used in lifetime are presented in section 3. Finally, Monte Carlo simulation to estimation of generalized Pareto distribution based on TL-moment with different trimming is illustrated, as an application, in section 4.

2. TL-MOMENTS OF RESIDUAL LIFE

Nair and Sankaran (2009) have defined the basic reliability functions such as failure rate, mean residual life, etc. in terms of quantile functions and presented several identities connecting them. Let X be a continuous nonnegative random variable representing lifetime with continuous distribution function $F(x)$ satisfying $F(0-) = 0$ and quantile function

$$Q(u) = \inf\{F(x) \geq u\}, \quad 0 < u < 1.$$

Then the mean residual life $m(t) = E(X - t|X > t) = E(X_{1:1} - t|X > t)$ is

$$m(t) = E(X - t|X > t) = \frac{1}{\bar{F}(t)} \int_t^\infty (x - t)f(x)dx = \frac{1}{\bar{F}(t)} \int_t^\infty xf(x)dx - t$$

and variance residual life is

$$\begin{aligned} \sigma^2(t) &= E((X - t)^2|X > t) - E^2(X - t|X > t) \\ &= E((X_{1:1} - t)^2|X > t) - E^2(X_{1:1} - t|X > t) \\ &= \frac{1}{\bar{F}(t)} \int_t^\infty (x - t)^2 f(x) dx - m^2(t) \\ &= \frac{1}{\bar{F}(t)} \int_t^\infty x^2 f(x) dx - \left[\frac{1}{\bar{F}(t)} \int_t^\infty x f(x) dx \right]^2 \\ &= \frac{1}{\bar{F}(t)} \int_t^\infty x^2 f(x) dx - (m(t) + t)^2 \end{aligned} \tag{2}$$

Analogous results mean residual quantile function is

$$M(u) = (1 - u)^{-1} \int_u^1 (Q(p) - Q(u)) dp = (1 - u)^{-1} \int_u^1 Q(p) dp - Q(u)$$

Where $F(x) = p$ and $t = u$ and variance residual quantile function is

$$\sigma^2(u) = (1 - u)^{-1} \int_u^1 Q^2(p) dp - (M(u) + Q(u))^2$$

Likewise, the functions in reversed time (i.e. $X \leq t$) are, the reversed mean residual quantile function and reversed variance residual quantile function defined as

$$R(u) = u^{-1} \int_0^u (Q(u) - Q(p)) dp$$

and

$$D(u) = u^{-1} \int_0^u Q^2(p) dp - (Q(u) - R(u))^2,$$

Now, TL-moments $\lambda_r^{(t_1, t_2)}$ of order r is defined as

$$\begin{aligned} \lambda_r^{(t_1, t_2)} &= r^{-1} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} E(X_{r+t_1-j:r+t_1+t_2}) \\ &= r^{-1} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \frac{(r+t_1+t_2)!}{(r+t_1-j-1)!(t_2+j)!} \\ &\quad \times \int_0^\infty x F(x)^{r+t_1-j-1} (1 - F(x))^{t_2+j} f(x) dx, \end{aligned}$$

Where $r = 1, 2, \dots, t_1, t_2 = 0, 1, 2, \dots$, and

$E(X_{r:n}) = n \binom{n-1}{r-1} \int_0^\infty x F(x)^{r-1} (1 - F(x))^{n-r} f(x) dx$, is the r th order statistic in a sample of size n from distribution function is $F(x)$ and $f(x)$ is the density function of X .

In case the right trimming, where $t_1 = 0, t_2 = s$

$$\begin{aligned} \lambda_r^{(0, s)} &= r^{-1} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} E(X_{r-j:r+s}) \\ &= r^{-1} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \frac{(r+s)!}{(r-j-1)!(s+j)!} \end{aligned}$$



$$\times \int_0^{\infty} xF(x)^{r-j-1} (1 - F(x))^{t_2+j} f(x) dx,$$

The truncated variable $X_t = X(t) = (X|X > t)$ has survival function $\bar{F}_t(x) = \bar{F}(x)/\bar{F}(t)$ so that

$$\begin{aligned} \lambda_r^{(0,s)}(t) &= r^{-1} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} EX_{r-j:r+s}(t) \\ &= r^{-1} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \frac{(r+s)!}{(r-j-1)!(s+j)!} \\ &\times \int_t^{\infty} x \left(\frac{\bar{F}(t) - \bar{F}(x)}{\bar{F}(t)} \right)^{r-j-1} \left(\frac{\bar{F}(x)}{\bar{F}(t)} \right)^{s+j} \frac{f(x)}{\bar{F}(t)} dx, \end{aligned}$$

where,

$$\begin{aligned} EX_{r:n}(t) &= n \binom{n-1}{r-1} \int_t^{\infty} x \left(\frac{\bar{F}(t) - \bar{F}(x)}{\bar{F}(t)} \right)^{r-1} \left(\frac{\bar{F}(x)}{\bar{F}(t)} \right)^{n-r} \frac{f(x)}{\bar{F}(t)} dx, \end{aligned}$$

is the truncated r th order statistic in a sample of size n from distribution function is $F(x)$ and $f(x)$ is the density function of X .

The first two population TL-moments are given as

$$\lambda_1^{(0,s)}(t) = EX_{1:s+1}(t) = \frac{(s+1)}{\bar{F}(t)^{s+1}} \int_t^{\infty} x \bar{F}(x)^s f(x) dx$$

and

$$\begin{aligned} \lambda_2^{(0,s)}(t) &= \frac{1}{2} (EX_{2:s+2}(t) - EX_{1:s+2}(t)) \\ &= \frac{(s+2)}{2\bar{F}(t)^{s+2}} \int_t^{\infty} x \bar{F}(x)^s [(s+2)F(x) - (s+1)F(t) - 1] f(x) dx \end{aligned}$$

Thus the trimmed mean residual life is given as

$$m^{(0,s)}(t) = E(X_{1:s+1} - t | X > t) = \frac{(s+1)}{\bar{F}(t)^{s+1}} \int_t^{\infty} x \bar{F}(x)^s f(x) dx - t$$

Also the trimmed variance residual life is given as

$$\begin{aligned} \sigma^{2(0,s)}(t) &= E((X_{1:s+1} - t)^2 | X > t) \\ &= E^2(X_{1:s+1} - t | X > t) \\ &= \frac{(s+1)}{\bar{F}(t)^{s+1}} \int_t^{\infty} (x-t)^2 \bar{F}(x)^s f(x) dx \\ &= m^{2(0,s)}(t) \\ &= \frac{(s+1)}{\bar{F}(t)^{s+1}} \int_t^{\infty} x^2 \bar{F}(x)^s f(x) dx \\ &- \left[\frac{(s+1)}{\bar{F}(t)^{s+1}} \int_t^{\infty} x \bar{F}(x)^s f(x) dx \right]^2 \\ &= \frac{(s+1)}{\bar{F}(t)^{s+1}} \int_t^{\infty} x^2 \bar{F}(x)^s f(x) dx \\ &- (m^{(0,s)}(t) + t)^2 \end{aligned}$$

Setting $F(x) = p$ and $F(t) = u$, we get the expression for the r th TL-moment residual quantile function of X as

$$\begin{aligned} \alpha_r^{(0,s)}(u) &= r^{-1} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \xi_{r-j:r+s} \\ &= \frac{(r+s)!}{r(1-u)^{r+s}} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \frac{1}{(r-j-1)!(s+j)!} \\ &\times \int_u^1 (p-u)^{r-j-1} (1-p)^{s+j} Q(p) dp, \end{aligned}$$

where

$$\begin{aligned} \xi_{r:n}(u) &= EX_{r:n}(u) = \\ &n \binom{n-1}{r-1} \int_u^1 \left(\frac{p-u}{1-u} \right)^{r-1} \left(\frac{1-p}{1-u} \right)^{n-r} \frac{Q(p)}{1-u} dp, \end{aligned}$$

is the truncated r th order statistic in a sample of size n as a function of $Q(u)$.

The first two population TL-moments are given as

$$\alpha_1^{(0,s)}(u) = \frac{(s+1)}{(1-u)^{s+1}} \int_u^1 (1-p)^s Q(p) dp \tag{15}$$

$$\begin{aligned} \alpha_2^{(0,s)}(u) &= \frac{(s+2)}{2(1-u)^{s+2}} \int_u^1 (1-p)^s [(s+2)p - (s+1)u - 1] Q(p) dp \\ &= \frac{(s+2)}{2(1-u)^{s+2}} \int_u^1 (1-p)^{s+1} (p - u) Q(p) dp \end{aligned}$$



The trimmed mean residual life quantile function

$$\begin{aligned}
 M^{(0,s)}(u) &= \frac{(s+1)}{(1-u)^{s+1}} \int_u^1 (1-p)^s (Q(p) - Q(u)) dp \\
 &= \frac{(s+1)}{(1-u)^{s+1}} \int_u^1 (1-p)^s Q(p) dp \\
 &\quad - Q(u) = \alpha_1^{(0,s)}(u) - Q(u) \\
 &= \frac{1}{(1-u)^{s+1}} \int_u^1 (1-p)^{s+1} \dot{Q}(p) dp
 \end{aligned}$$

By derivative of $\alpha_1^{(0,s)}(u)$ we obtain

$$Q(u) = \alpha_1^{(0,s)}(u) - \frac{(1-u)}{(s+1)} \dot{\alpha}_1^{(0,s)}(u)$$

By derivative of $M^{(0,s)}(u)$ we obtain

$$\begin{aligned}
 Q(u) &= \mu^{(0,s)} - M^{(0,s)}(u) \\
 &\quad + (s+1) \int_0^u \frac{M^{(0,s)}(p)}{(1-p)} dp \quad Q(u)
 \end{aligned}$$

Where

$$\mu^{(0,s)} = M^{(0,s)}(0) = \alpha_1^{(0,s)}(0) = (s+1) \int_0^1 (1-u)^s Q(u) du$$

is the TL- mean residual life quantile function .

Also, the TL-variance residual life quantile function

$$\begin{aligned}
 \sigma^{2(0,s)}(u) &= \frac{(s+1)}{(1-u)^{s+1}} \int_u^1 (1-p)^s Q^2(p) dp \\
 &\quad - \left(M^{(0,s)}(p) + Q(u) \right)^2 \\
 &= \frac{(s+1)}{(1-u)^{s+1}} \int_u^1 (1-p)^s Q^2(p) dp \\
 &\quad - \left(\alpha_1^{(0,s)}(u) \right)^2
 \end{aligned}$$

By obtain $\frac{d}{du} (1-u)^{s+1} \sigma^{2(0,s)}(u)$ we get

$$\sigma^{2(0,s)}(u) = \frac{(s+1)}{(1-u)^{s+1}} \int_u^1 (1-p)^s \left(M^{(0,s)}(p) \right)^2 dp$$

Note: In particular, when $s = 0$ the TL-moments reduced to L-moment; see Nair and Vineshkumar (2010).

Theorem 1: The function $\alpha_1^{(0,s)}(u), \alpha_2^{(0,s)}(u)$ and $M^{(0,s)}(u)$ determine each other and $Q(u)$ uniquely.

Proof: $M^{(0,s)}(u) = \alpha_1^{(0,s)}(u) - Q(u)$ the unique determination of $M^{(0,s)}(u)$ from $\alpha_1^{(0,s)}(u)$ and conversely follows from $M^{(0,s)}(u)$ and $Q(u)$.

Differentiating $\alpha_2^{(0,s)}(u)$ and simplifying

$$M^{(0,s)}(u) = 2 \alpha_2^{(0,s)}(u) - \frac{2(1-u)}{s+2} \dot{\alpha}_2^{(0,s)}(u)$$

Showing that $\alpha_2^{(0,s)}(u)$ determines $M^{(0,s)}(u)$ and hence $Q(u)$ uniquely. Also from $\alpha_1^{(0,s)}(u)$ we get

$$\frac{d}{du} \frac{2(1-u)^{s+2}}{(s+2)} \alpha_2^{(0,s)}(u) = -(1-u)^{s+1} M^{(0,s)}(u)$$

We have

$$\alpha_2^{(0,s)}(u) = \frac{(s+2)}{2(1-u)^{s+2}} \int_u^1 (1-p)^{s+1} M^{(0,s)}(p) dp \tag{24}$$

Giving $\alpha_2^{(0,s)}(u)$ in terms of $M^{(0,s)}(u)$.

Remark 1: Using distribution functions, the analogous results are

$$\begin{aligned}
 \lambda_2^{(0,s)}(t) &= \frac{(s+2)}{2\bar{F}(t)^{s+2}} \int_t^\infty \bar{F}(x)^s ((s+1)\bar{F}(t) \\
 &\quad - (s+2)\bar{F}(x)) x f(x) dx \\
 &= \frac{(s+2)}{2} \left[\lambda_1^{(0,s)}(t) - t \right. \\
 &\quad \left. - \frac{1}{\bar{F}(t)^{s+2}} \int_t^\infty \bar{F}(x)^{s+2} dx \right] \\
 &= \frac{(s+2)}{2} \left[m^{(0,s)}(t) \right. \\
 &\quad \left. - \frac{1}{\bar{F}(t)^{s+2}} \int_t^\infty \bar{F}(x)^{s+2} dx \right]
 \end{aligned}$$

Differentiating the last expression

$$\dot{\lambda}_2^{(0,s)}(t) = \frac{(s+2)}{2} h(t) [2\lambda_2^{(0,s)}(t) - m^{(0,s)}(t)]$$

where, $h(t)$ is the failure rate function of X . This expression is important in deducing the conditions for the monotonic behaviour of $\lambda_2^{(0,s)}(t)$ when $F(x)$ is used instead of $Q(u)$

Since $\alpha_2^{(0,s)}(u)$ is conceived as a measure of dispersion its relationship with the trimmed variance residual quantile function is of interest. In fact from $\alpha_2^{(0,s)}(u)$ we have



$$\begin{aligned} \sigma^{2(0,s)}(u) &= \frac{(s+1)}{(1-u)^{s+1}} \int_u^1 (1-p)^s (M^{(0,s)}(p))^2 dp \\ &= \frac{(s+1)}{(1-u)^{s+1}} \int_u^1 (1-p)^s \left(2\alpha_2^{(0,s)}(p) - \frac{2(1-u)}{(s+2)} \alpha_2^{(0,s)}(p) \right)^2 \end{aligned}$$

Gupta and Kirmani (2000) have shown that the coefficient of variation of residual life characterizes the life distribution. We now demonstrate that a similar result exists for the TL-coefficient of variation of the residual quantile function defined as

$$c^{(0,s)}(u) = \alpha_2^{(0,s)}(u) / \alpha_1^{(0,s)}(u)$$

Theorem 2: If $c^{(0,s)}(u)$ is differentiable, then

$$Q(u) = (1-u)^{-s} g^{(0,s)}(u) \exp[-\int g^{(0,s)}(u) du]$$

where

$$g^{(0,s)}(u) = \frac{(1-u)\dot{c}^{(0,s)}(u) - c^{(0,s)}(u) + \frac{(s+2)}{2}}{(1-u) \left[\frac{(s+2)}{2(s+1)} + c^{(0,s)}(u) \right]}$$

Proof: From the definition of $c^{(0,s)}(u)$, $\alpha_1^{(0,s)}(u)$ and $\alpha_2^{(0,s)}(u)$,

$$\begin{aligned} (1-u)c^{(0,s)}(u) \int_u^1 (1-p)^s Q(p) dp &= \frac{(s+2)}{2(s+1)} \int_u^1 (1-p)^s ((s+2)p - (s+1)u - 1) Q(p) dp \end{aligned}$$

Differentiating and rearranging the terms

$$\begin{aligned} \frac{\int_u^1 (1-p)^s Q(p) dp}{(1-u)^s Q(u)} &= \frac{(1-u)\dot{c}^{(0,s)}(u) - c^{(0,s)}(u) + \frac{(s+2)}{2}}{(1-u) \left[\frac{(s+2)}{2(s+1)} + c^{(0,s)}(u) \right]} \end{aligned}$$

and integrating two sides

$$\begin{aligned} -\log_e \int_u^1 (1-p)^s Q(p) dp &= \int \frac{(1-u)\dot{c}^{(0,s)}(u) - c^{(0,s)}(u) + \frac{(s+2)}{2}}{(1-u) \left[\frac{(s+2)}{2(s+1)} + c^{(0,s)}(u) \right]} \end{aligned}$$

from which the equation follows.

3. QUANTILE FUNCTION MODEL

In this section In this section, we illustrate the potential of quantile functions based on TL-moments, as alternatives to distribution functions, as models of lifetime data by presenting a few simple forms. They all share the property of different distributional shapes that can either exactly or approximately represent real life data. Also they do not have distribution functions in simple closed forms to facilitate the use of conventional reliability concepts in the analysis.

3.1 Govindarajulu distribution

Govindarajulu (1977) has proposed a class of distributions with

$$Q(u) = \theta + \sigma[(\beta + 1)u^\beta - \beta u^{\beta+1}], \quad \beta > 0, \sigma > 0$$

for applications in life testing and reliability. He studied the properties of $X = \sigma^{-1}(X - \theta)$, discussed method of estimating the parameters and fitted the distribution to the data on failure times of a set of refrigerator motors.

For the random variable Z , where $\theta = 0$ and $\sigma = 1$, the first TL-moment, $\alpha_1^{(0,s)}(u)$, the second TL-moment, $\alpha_2^{(0,s)}(u)$ and T-mean $M_{(u)}^{(0,s)}$ of residual life given as

$$\begin{aligned} \alpha_1^{(0,s)}(u) &= \frac{(s+1)}{(1-u)^{s+1}} \sum_{k=0}^s (-1)^k \binom{s}{k} \left[\frac{(\beta+1)(1-u^{\beta+k+1})}{(\beta+k+1)} - \frac{\beta(1-u^{\beta+k+2})}{(\beta+k+2)} \right] \\ \alpha_2^{(0,s)}(u) &= \frac{(s+2)\beta(\beta+1)}{2(1-u)^{s+2}} \sum_{k=0}^{s+2} (-1)^k \binom{s+2}{k} \left[\frac{(1-u)^{-(\beta+k)}}{(\beta+k)(\beta+k+1)} - \frac{(1-u)^{\beta+k+1}}{(\beta+k)(\beta+k+1)} \right] \end{aligned}$$

and

$$M_{(u)}^{(0,s)} = \frac{\beta(\beta+1)}{(1-u)^{s+1}} \sum_{k=0}^{s+2} (-1)^k \binom{s+2}{k} \frac{(1-u)^{\beta+k}}{(\beta+k)}$$

where $s = 0$: the first TL-moment, $\alpha_1^{(0,s)}(u)$, the second TL-moment, $\alpha_2^{(0,s)}(u)$ and T-mean $M_{(u)}^{(0,s)}$ of residual life become

$$\begin{aligned} \alpha_1^{(0,0)}(u) &= \frac{2 - (\beta + 2)u^{\beta+1} + \beta u^{\beta+2}}{(\beta + 2)(1 - u)} \\ \alpha_2^{(0,0)}(u) &= \frac{1}{(\beta + 2)(\beta + 3)(1 - u)^2} [2\beta - 2(\beta + 3)u + (\beta + 2)(\beta + 3)u^{\beta+1} - 2\beta(\beta + 3)u^{\beta+2} + \beta(\beta + 1)u^{\beta+3}] \end{aligned}$$

and



$$M_{(u)}^{(0,0)} = \frac{12 - (\beta + 1)(\beta + 2)u^\beta + 2\beta(\beta + 2)u^{\beta+1} - \beta(\beta + 1)u^\beta}{(\beta + 2)(1 - u)}$$

While, where $s = 1$: the first TL-moment, $\alpha_1^{(0,s)}(u)$, the second TL-moment, $\alpha_2^{(0,s)}(u)$ and T-mean $M_{(u)}^{(0,s)}$ of residual life become

$$\alpha_1^{(0,1)}(u) = \frac{6 - 2(\beta + 2)(\beta + 3)u^{\beta+1} + 2(2\beta + 1)(\beta + 3)u^{\beta+2} - 2\beta(\beta + 2)u^{\beta+3}}{(\beta + 2)(\beta + 3)(1 - u)^2}$$

$$\alpha_2^{(0,1)}(u) = \frac{2(\beta + 2)(\beta + 3)(\beta + 4)(1 - u)^3 [2\beta - 6(\beta + 4)u + (\beta + 2)(\beta + 3)(\beta + 4)u^{b+1} - 3\beta(\beta + 3)(\beta + 4)u^{\beta+2} + 3\beta(\beta + 1)(\beta + 4)u^{b+3} - \beta(\beta + 1)(\beta + 2)u^{\beta+4}]}{3}$$

and

$$M_{(u)}^{(0,1)} = \frac{1}{(\beta + 2)(\beta + 3)(1 - u)^2} [6 - (\beta + 1)(\beta + 2)(\beta + 3)u^\beta + 3\beta(\beta + 2)(\beta + 3)u^{\beta+1} - 3\beta(\beta + 1)(\beta + 3)u^{\beta+2} + \beta(\beta + 1)(\beta + 2)u^{\beta+3}]$$

In Fig.1(a): indicate that both functions, $M_{(u)}^{(0,s)}$ and $\alpha_2^{(0,s)}(u)$, decrease for all u and at $s = 0$ and $s = 1$. In Fig. (b) indicate that $M_{(u)}^{(0,s)}$, at $s = 0$, decreases for all u . While, at $s = 1$, initially increases, constant in short interval and then decreases. For $\alpha_2^{(0,s)}(u)$, at $s = 0$, decreases for all u . But at $s = 1$, $\alpha_2^{(0,s)}(u)$ initially constant in short interval and then decreases. In Fig.1(c): indicate that $M_{(u)}^{(0,s)}$, at $s = 0$, initially increases, constant in short interval, approximately, $(0.2556 \leq u \leq 0.2961)$ and then decreases. Also, at $s = 1$, initially increases, constant in interval, approximately, $(0.2975 \leq u \leq 0.3910)$ and then decreases. While, $\alpha_2^{(0,s)}(u)$, at $s = 0$, decreases for all u . But at $s = 1$, $\alpha_2^{(0,s)}(u)$ initially increases, constant in short interval, approximately, $(0.1289 \leq u \leq 0.2278)$ and then decreases. In Fig.1(d): indicate that both functions, $M_{(u)}^{(0,s)}$ and $\alpha_2^{(0,s)}(u)$, initially increase, constant in short interval and then decrease for all $s = 0$ and $s = 1$.

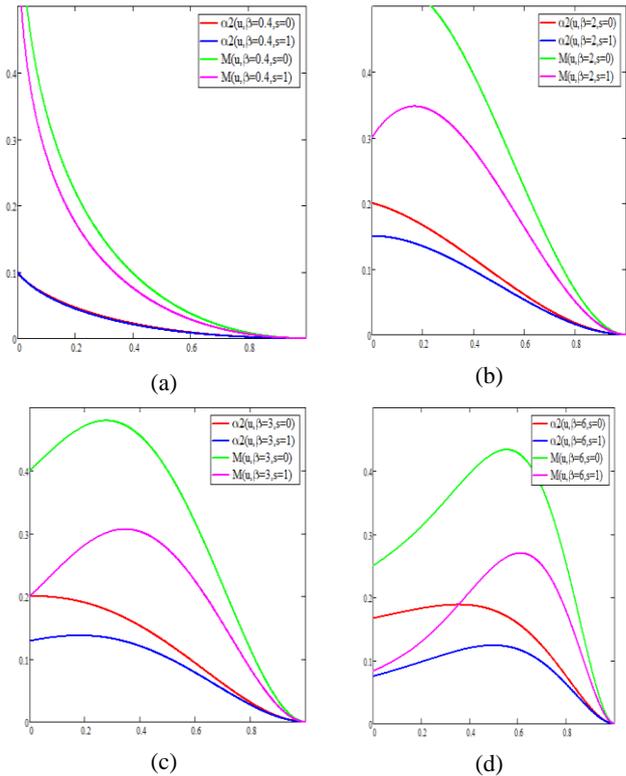


Figure 1: The second TL-moment, $\alpha_2^{(0,s)}(u)$ and T-mean $M_{(u)}^{(0,s)}$ of residual life for Govindarajulu distribution with different value of the parameter and trim values $s=0$ and 1 .

3.2 The generalized Pareto distribution

The generalized Pareto distribution (GPD) has quantile function

$$Q(u) = \frac{b}{a} \left[(1 - u)^{\frac{-a}{a+1}} - 1 \right] \quad a > -1, b > 0$$

which is a family consisting of the Exponential distribution ($a \rightarrow 0$), rescaled beta ($-1 < a < 0$) and the Lomax distribution ($a > 0$). The family is characterized by a linear mean residual life (reciprocal linear hazard rate) function in the conventional reliability analysis.

The first TL-moment, $\alpha_1^{(0,s)}(u)$, the second TL-moment, $\alpha_2^{(0,s)}(u)$ and T-mean $M_{(u)}^{(0,s)}$ of residual life given as

$$\alpha_1^{(0,s)}(u) = \frac{b}{a} \left(\frac{(s+1)(a+1)(1-u)^{\frac{-a}{a+1}}}{s(a+1)+1} - 1 \right)$$

$$\alpha_2^{(0,s)}(u) = \frac{b(s+2)(a+1)(1-u)^{\frac{-a}{a+1}}}{2(s(a+1)+(a+2))(s(a+1)+1)}$$

and

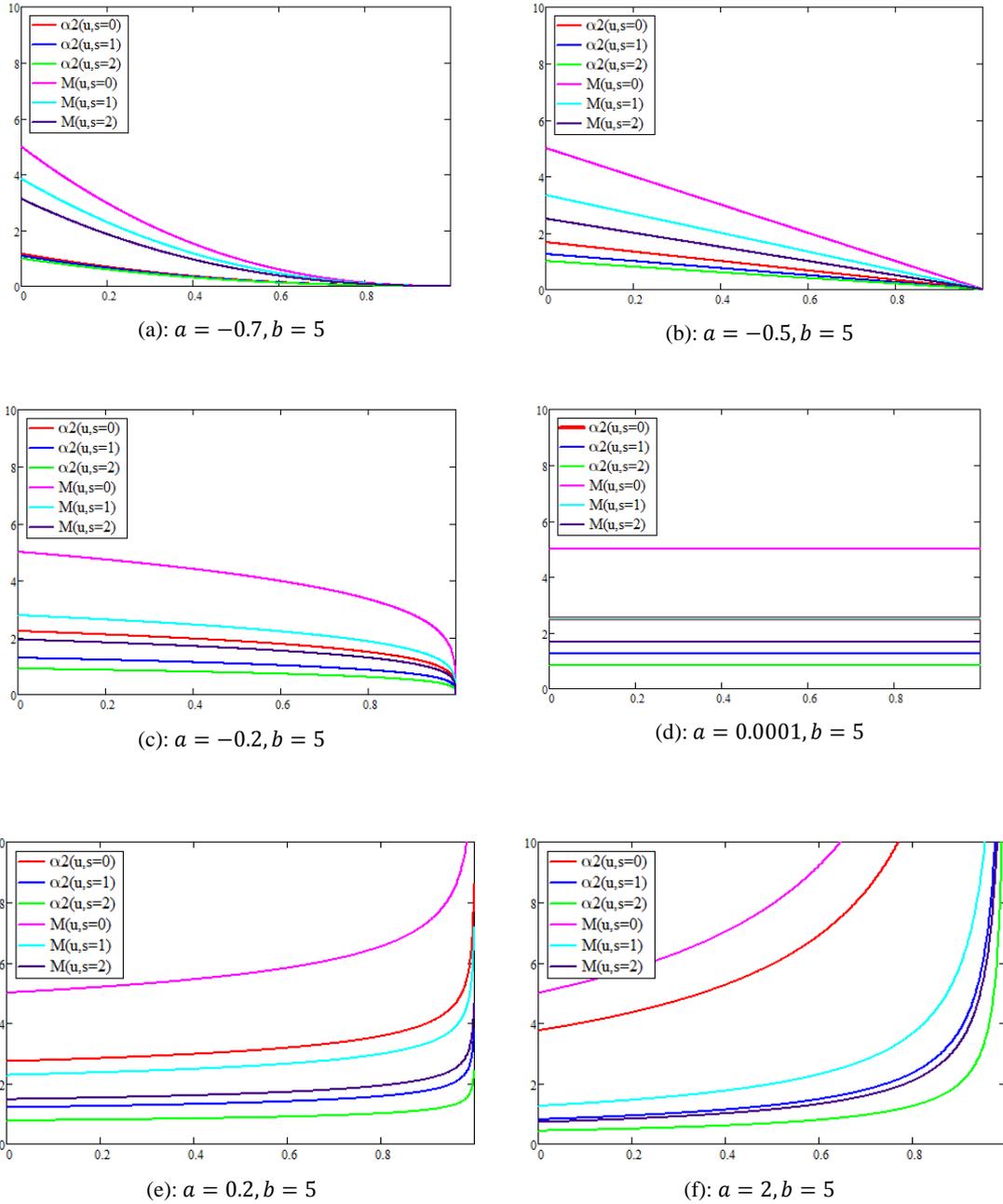


Figure 2: The second TL-moment, $\alpha_2^{(0,s)}(u)$ and T-mean $M_{(u)}^{(0,s)}$ of residual life for generalized Pareto distribution with different values of shape parameter and trimming $s=0,1$ and 2 .

$$M^{(0,s)}(u) = \frac{b(1-u)^{\frac{-a}{a+1}}}{s(a+1)+1} = \mu^{(0,s)} (1-u)^{\frac{-a}{a+1}}$$

IFigure 2 indicates that both functions, $M_{(u)}^{(0,s)}$ and $\alpha_2^{(0,s)}(u)$, with all trimming values $s=0,1$ and 2 have

the same behavior. Decrease in interval $(-1 < a < 0)$, constant ($a \rightarrow 0$) and increase in interval $(a > 0)$ for all u .



4. APPLICATION

In this section we introduce Monte Carlo simulation to estimation of generalized Pareto distribution based on TL-moment with different trimming value.

For estimating the parameters using the method of TL-moments, the sample TL-moments (right trimming) are computed as

$$l_{r+1}^{(0,s)} = \frac{1}{(r+1) \binom{n}{s+r+1}} \sum_{j=0}^r (-1)^j \binom{r}{j} \sum_{i=0}^n \binom{i-1}{r-j} \binom{n-i}{s+j} x_{i:n},$$

$r = 0, 1, \dots, n, s = 0, 1, \dots, n - 1.$

Where $x_{i:n}$ is the i th order statistic in a sample of size n . By equating the first two population TL-moments with the corresponding sample TL-moments for generalized Pareto distribution, we obtain the following equations

$$\frac{b}{s(a+1)+1} = l_1^{(0,s)}$$

and

$$\frac{b(s+2)(a+1)}{2(s(a+1) + (a+2))(s(a+1) + 1)} = l_2^{(0,s)}$$

Solving above equations give

$$\hat{a} = \frac{(s+2)(l_1^{(0,s)} - 2l_2^{(0,s)})}{2(s+1)l_2^{(0,s)} - (s+2)l_1^{(0,s)}}$$

and

$$\hat{b} = (s(\hat{a} + 1) + 1)l_1^{(0,s)}$$

For TL-moment estimators with different trimming values bias and mean square error of estimates of the scale and shape parameters were obtained using Monte Carlo simulation. Tables 1, 2 and 3 shows the results for sample sizes 25, 50, 100 and 200 over a range of positive values of a . As a increases and the tail weight of the distribution increases, larger amount of trimming become preferable: where they have lowest mean square error. This is also true for b .

Table 1: Simulated biased and mean square error (MSE) of the shape and scale parameters ($a = 0.25, b = 1$) from Pareto distribution using TL-moment estimator with different trimming and number of replication is 5000.

$a = 0.25$				
Sample sizes				
	25	50	100	200
(0,s)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)
(0,0)	-1.44(0.0871)	0.110(0.0420)	0.058(0.0199)	0.033(0.0100)
(0,1)	-1.95(0.1679)	-0.17(0.0491)	-0.009(0.0264)	0.003(0.0129)
(0,2)	-2.18(0.3263)	-0.009(0.1014)	-0.004(0.0492)	0.008(0.0256)
$b = 1$				
(0,0)	0.120(0.1906)	0.149(0.1091)	0.079(0.0385)	0.033(0.0193)
(0,1)	0.065(0.1942)	0.019(0.0751)	0.013(0.0338)	0.003(0.0168)
(0,2)	0.055(0.2398)	0.026(0.0876)	0.017(0.0409)	0.007(0.0197)

Table 2: Simulated biased and mean square error (MSE) of the shape and scale parameters ($a = 0.75, b = 1$) from Pareto distribution using TL-moment estimator with different trimming and number of replication is 5000.

$a=0.75$				
Sample sizes				
	25	50	100	200
(α, β)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)
(0,0)	0.385(0.2164)	0.242(0.0948)	0.168(0.0520)	0.115(0.0287)
(0,1)	-0.020(0.1504)	-0.053(0.0826)	-0.045(0.0436)	-0.031(0.0239)
(0,2)	-0.052(0.2519)	-0.033(0.1189)	-0.022(0.0523)	-0.006(0.0268)
$b = 1$				
(0,0)	0.899(2.9800)	0.537(0.7273)	0.302(0.1679)	0.196(0.1698)
(0,1)	0.117(0.2603)	0.042(0.1022)	-0.001(0.0488)	-0.011(0.0255)
(0,2)	0.086(0.2730)	0.053(0.1126)	0.012(0.0463)	0.005(0.0232)



Table 3: Simulated biased and mean square error (MSE) of the shape and scale parameters ($a = 1.25, b = 1$) from Pareto distribution using TL-moment estimator with different trimming and number of replication is 5000.

		$a = 1.25$			
		Sample sizes			
		25	50	100	200
α, β	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)
(0,0)	0.572(0.3793)	0.475(0.2508)	0.427(0.1976)	0.401(0.1695)	
(0,1)	0.245(0.1926)	0.144(0.0999)	0.084(0.0545)	-0.058(0.0553)	
(0,2)	0.136(0.2642)	0.081(0.1271)	0.028(0.0585)	-0.039(-0.0440)	
		$b = 1$			
	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)
(0,0)	1.239(5.7847)	1.024(2.3187)	0.949(2.1482)	1.86(33.015)	
(0,1)	0.254(0.5118)	0.129(0.1790)	0.078(0.0841)	-0.001(0.0525)	
(0,2)	0.118(0.3871)	0.054(0.1381)	0.016(0.0595)	0.008(0.0342)	

REFERENCES

[1] Abdul-Moniem, I. B., 2007. L -Moments and TL - Moments Estimation for the Exponential Distribution. *Far East Journal of Theoretical Statistics*, **23**, 51-61.

[2] Abdul-Moniem, I. B. and Selim, Y. M., 2009. TL-moments and L-moments Estimation for the Generalized Pareto Distribution. *Applied Mathematical Sciences*, **3**, no. (1), 43-52.

[3] Asquith, W. H., 2007. L-moments and TL-moments of the generalized lambda distributions. *Computational Statistics & Data Analysis*, **51**, 4484 - 4496.

[4] Elamir, E. A. H. and Seheult, A. H., 2003. Trimmed L-moments. *Computational Statistics Data Analysis*, **43**, 299-314.

[5] Elamir, E. A. H., 2009. Nonparametric Estimation for Quantile and Sparsity Functions via Trimmed L-moments. *Essays on Mathematics and Statistics, Edited by Vladimir Akis, published by ATINER*, 195-210.

[6] Elamir, E. A. H., 2010. Optimal Choices for Trimming in Trimmed L-moment Method. *Applied Mathematical Sciences*, **4**, 2881 – 2890.

[7] Govindarajulu, Z., 1977. A class of distributions useful in life testing and reliability with applications to nonparametric testing. In: Tsokos, C.P., Shimi, I.N. (Eds.), *The Theory and Applications of Reliability*, Academic Press, New York, vol. I, 109–130.

[8] Gupta, R.C., Kirmani, S.N.U.A., 2000. Residual coefficient of variation and some characterization results. *Journal of Statistical Planning and Inference*, **91**, 23–31.

[9] Gupta, R.C., Kirmani, S.N.U.A., 2004. Some characterizations of distributions by functions of failure rates and mean residual life. *Communications in Statistics: Theory and Methods* **33**, 3115–3131.

[10] Hosking, J. R. M., 1990. L-moments: analysis and estimation of distributions using linear combinations of statistics. *Journal of the Royal Statistical Society, Ser. B*, **52**, 105-124.

[11] Hosking, J. R. M., 1992. Moments or L-moments? An example comparing two measures of distributional shape. *The American Statistician*, **46**, 186-189.

[12] Hosking, J.R.M., 2007. Some theory and practical uses of trimmed L-moments. *Journal of Statistical Planning and Inference*. **137**, 3024-3039.

[13] Hosking, J. R. M. and Wallis, J. R., 1997. *Regional frequency analysis: An approach based on L-moments*. University press, Cambridge.

[14] Karvanen, J., 2006. Estimation of quantile mixtures via L-moments and trimmed L-moments. *Computational Statistics & Data Analysis*, **51**, 947–959.

[15] Maillet, B. and Me´decin J. P., 2010. Financial crises, extreme volatilities and L-moment estimations of tail-indexes. Available at: <http://ssrn.com/abstract=1609573>

[16] Nair, N. U. and Sankaran, P. G., 2009. Quantile based reliability analysis. *Communications in Statistics Theory and Methods*, **38**, 222-232.

[17] Nair, N. U. and Vineshkumar, B., 2010. L-moments of residual life. *Journal of Statistical Planning and Inference*, **140**, 2618–2631.

[18] Perez, N. H., Manendez, S. C., Seco, L., 2003. A theoretical comparison between moments and L-moments. Available at: <http://citeseerx.ist.psu.edu/viewdoc/versions?doi:10.1.1.33.0.8640>.

[19] Sankarasubramanian, A. and Srinivasan, K., 1999. Investigation and comparison of sampling properties of L-moments and conventional moments. *Journal of Hydrology*, **218**, 13–34.