International Journal of Computational and Theoretical Statistics
ISSN (2210-1519)
Int. J. Comp. Theo. Stat. 2, No. 2 (Nov-2015)

Test for Accuracy of Ranking in Moving Extreme Ranked Set Sampling

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Received 21 January 2015, Revised 3 April 2015, Accepted 28 April 2015, Published 1 November 2015

Abstract: Moving Extreme Ranked Set Sampling (MERSS) is a variation of Ranked Set Sampling (RSS) that simplifies the technique and makes it more applicable. In MERSS, the judgment maximum of random samples of sizes 1, 2,…, are taken for actual measurement. Testing for error in ranking should be done before using the MERSS for inference. Testing whether judgment ranking is as good as actual ranking is considered in this paper. Three nonparametric tests are considered. These tests are mainly based on the distance between the actual and the judgment ranking of the obtained data. The null and the alternative distributions of the test statistics are derived. A real data set is used for illustration.

Keywords: Ranked Set Sampling; Moving Extreme Ranked Set Sampling; Concomitant Order Statistic; Error in Ranking.

1. Introduction

Ranked set sampling (RSS) technique was introduced by McIntyre (1952) to estimate more effectively yields of pastures. This method of selecting a sample is suitable for situations when the units can be ranked (with respect to the variable of interest) by judgment without actual measurement. The main idea of RSS is similar to stratified sampling; in stratified sampling, the population is divided by judgment into sub populations (strata) so that elements are more similar within strata than among strata. In ranked set sampling, we are trying to do the same as in stratified sampling but at the level of the sample rather than the level of the population. The RSS technique can be executed as follows:

1. m sets of size m each are drawn randomly from the population of interest;
2. The elements within each chosen set are ranked by judgment (without doing actual quantification) from smallest to largest with respect to the variable of interest;
3. From the i^{th} set, the element (judgment) ranked as the i^{th} order statistic is take for actual quantification, $i = 1,2,\dots,m$.

   This cycle (steps 1,2,3) yields a ranked set sample of size m.

4. The above procedure can be repeated r times to get a sample of size $n = rm$.
   The main statistical theory of RSS was developed by Takahasi and Wakimoto (1968). They showed that the mean of RSS is the best linear unbiased estimator of the population mean and is more efficient than the mean of SRS with the same size. For more details and results on RSS technique see Kaur et al. (1995), and Chen et al. (2004). Al-Saleh and Zheng(2002); Zheng and Al-Saleh(2002); Al-Saleh and Al-Omary(2002).

Tests for perfect ranking in RSS was considered by Li and Balakrishnan (2008) and Vock and Balakrishnan (2011). They proposed nonparametric tests for testing the assumption of perfect ranking. Their proposed tests were based on the probability of $P(Y_{i_1} \le Y_{i_2} \le \dots \le Y_{i_m})$, where Y_{i_j} are elements of RSS of size m; a closed formula for this probability was obtained by Al-Saleh and Al-Kadiri (2000). Also, other tests were proposed by Li and Balakrishnan (2008) based on multi-cycle RSS. In addition, Vock and Balakrishnan (2011) used the test statistic that formally corresponds to the Jonckheere-Terpstra-type test.

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Al-Odat and Al-Saleh (2001) introduced a new modified technique of RSS; later it was given the name **"Moving Extreme Ranked Set Sampling" (MERSS)** by Al-Saleh and Al-Hadhrami (2003). They showed that this modification of RSS can be more useful than SRS and easier to perform. They investigated this method nonparametrically and concluded that the estimator of the population mean is more efficient than that of SRS in the case of symmetric populations. The method was considered parametrically under exponential distribution by Al-Saleh and Al-Hadhrami (2003 a, b); they studied this method in case of perfect and imperfect ranking. Also, the maximum likelihood estimator (MLE) and modified MLE of the population mean were considered. Al-Saleh and Al-Ananbeh (2005) considered the estimation of correlation coefficient in the bivariate normal distribution based on MERSS using a concomitant random variable. Al-Saleh and Al-Ananbeh (2007) considered the estimation of the means of the bivariate normal distribution based on MERSS with concomitant variable. Abu-Dayyeh and Al-Sawi (2009) made inference about the scale parameter of the exponential density in the case of MERSS using the maximum likelihood estimator and the likelihood ratio test (LRT). Al-Saleh and Samawi (2010, 2011) used MERSS to estimate the odds and odds ratio. Inference on Downton's Bivariate Exponential Distribution Based on MERSS was considered by Hannadeh and Al-Saleh (2013).
The MERSS technique can be described as follows:

1) Select m simple random samples of size $1,2,\ldots,m$, respectively;
2) The maximum of each of the m sets is identified and quantified. These maxima should be identified by judgment or by costless method.
3) Steps (1 and 2) can be repeated, if necessary, many times to obtain a sample of larger size.

In this paper, the MERSS technique is investigated non-parametrically; i.e. there is no assumption about the underlying distribution. Let $Y_{[i_k:i_k]}$ is the judgment maximum order statistic for a sample of size i_k , $i_k k=1,2,\ldots,m$.
The probability

$$\pi(i_1,i_2,\ldots,i_m)=P\left(Y_{[i_1:i_1]}\le Y_{[i_2:i_2]}\le\ldots\le Y_{[i_m:i_m]}\right)$$

is derived under perfect and imperfect ranking, and some properties are listed and proved. Then, three simple non-parametric tests are investigated to test for perfect ranking using one-cycle MERSS. The exact null distributions of these tests are found, and the exact power functions under some specific alternatives are derived. Tests that deal with multi-cycle MERSS are introduced. Samples from bivariate normal distribution are used for illustrations.

2. Test for Perfect and Imperfect Ranking in MERSS-One Cycle

In this section, we consider the error in ranking in MERSS. Three statistical tests to test for imperfect ranking are discussed; they are denoted by N_m, S_m and A_m. The three tests are investigated based on one cycle MERSS.

Let $\{Y_{[i:i]}, i=1,2,\ldots,m\}$ be a MERSS of size m, where $Y_{[i:i]}$ is the judgment maximum order statistic of a SRS of size i with pdf f and cdf F; it is assumed to be absolutely continuous. Also, assume that $Y_{(i:i)}$ is the actual maximum order statistic of a SRS of size i. The probability density functions of $Y_{(i:i)}$ and $Y_{[i:i]}$ are, respectively:

$$f_{(i:i)}(y)=i\left(F(y)\right)^{i-1}f(y),\qquad f_{[i;i]}(y)=\sum_{k=1}^{i}a_{ki}\,f_{(k:i)}(y),\ \ i=1,2,\ldots,m.$$

Where,

$$f_{(k:i)}(y)=i\binom{i-1}{k-1}F^{k-1}(y)\left(1-F(y)\right)^{i-k}f(y),$$

a_{ki} are positive constants, $0\le a_{ki}\le 1$, $\sum_{k=1}^{i}a_{ki}=1$, a_{ki} can be thought of as the probability that the $Y_{[i:i]}$ has the density $f_{(k:i)}$. If the ranking is perfect, then $a_{ki}=1$ for $k=i$ and zero otherwise. (For more details, see Frey (2007)).
Our hypotheses are:
H_0: $a_{ki}=1$ for $k=i$, $i=1,2,\ldots,m$ and $a_{ki}=0$, otherwise; i.e. **Ranking is perfect (no ranking error)**
H_1**: Ranking is imperfect (there is some error in ranking).**

Let $\{Y_{[i:i]}, i = 1,2,\ldots,m\}$ be a MERSS of size m, let $(i_1, i_2, \ldots, i_m)$ be any permutation of $(1,2,\ldots,m)$, $\pi(i_1, i_2, \ldots, i_m) = P(Y_{[i_1:i_1]} \le Y_{[i_2:i_2]} \le \ldots \le Y_{[i_m:i_m]})$ under perfect ranking is given by the following theorem:

Theorem (1): For any permutation $(i_1, i_2, \ldots, i_m)$ of $(1,2,\ldots,m)$, if H_0 is true then:

$$\pi_0(i_1, i_2, \ldots, i_m) = P(Y_{[i_1:i_1]} \le Y_{[i_2:i_2]} \le \ldots \le Y_{[i_m:i_m]}) = \frac{m!}{\prod_{k=1}^{m} \sum_{j=1}^{k} i_j} = \frac{m!}{i_1(i_1+i_2)(i_1+i_2+i_3)\ldots(i_1+i_2+\ldots+i_m)}$$

The proof of the theorem is straight forward using the basic integration techniques.

The following are some properties of π_0 that can be easily verified:

1. For $i_1 < i_2$, $P(Y_{(i_1:i_1)} < Y_{(i_2:i_2)}) > 0.5$;
2. $\pi_0(1,2,\ldots,m) \ge \frac{1}{m!}$;
3. $\pi_0(1,2,\ldots,m) = \frac{2^m}{(m+1)!}$;
4. $\pi_0(i_1, i_2, \ldots, i_m) = \pi_0(1,2,\ldots,m)$, i.e. the maximum value of π_0 occurred at $(1,2,\ldots,m)$.
5. $\pi_0(i_1, i_2, \ldots, i_m) \ge \pi_0(m, m-1, \ldots, 2, 1) = \frac{2^m m!}{(2m)!}$, i.e. the minimum value of π_0 occurred at $(m, m-1, \ldots, 2, 1)$.

From 3 and 5 we have:

$$\frac{2^m m!}{(2m)!} \le \pi_0(i_1, i_2, \ldots, i_m) \le \frac{2^m}{(m+1)!}.$$

3. $\pi(i_1, i_2, \ldots, i_m)$ Under Imperfect Ranking

In the previous section, we found the value of $\pi(i_1, i_2, \ldots, i_m)$ under perfect ranking. Here, we assume that there is an error in judgment ranking; in this case we assume that the probability density function of $Y_{[i:i]}$ takes the form (Frey, 2007):

$$f_{[i;i]}(y) = \sum_{k=1}^{i} a_{ki} f_{(k:i)}(y), \quad i = 1,2,\ldots,m$$

where,

$$f_{(k:i)}(y) = i\binom{i-1}{k-1} F^{k-1}(y)(1-F(y))^{i-k} f(y), \; k = 1,2,\ldots,i \text{ and } \sum_{k=1}^{i} a_{ki} = 1.$$

Now,

$$\pi(i_1, i_2, \ldots, i_m) = P(Y_{[i_1:i_1]} \le Y_{[i_2:i_2]} \le \ldots \le Y_{[i_m:i_m]}) = \int_{-\infty}^{\infty} \int_{-\infty}^{y_{i_m}} \ldots \int_{-\infty}^{y_{i_3}} \int_{-\infty}^{y_{i_2}} \prod_{i=1}^{m} \sum a_{ki} f_{i_k}\left(y_{i_k}\right) dy_{i_k}$$

Let $F(y_{i_k}) = u_{i_k}$, then $f(y_{i_k}) dy_{i_k} = du_{i_k}$, thus,

$$\pi(i_1, i_2, \ldots, i_m) = \int_{0}^{1} \int_{-\infty}^{u_{i_m}} \ldots \int_{-\infty}^{u_{i_3}} \int_{-\infty}^{u_{i_2}} \prod_{i=1}^{m} \sum_{k=1}^{i} a_{ki}\, i\binom{i-1}{k-1} u_{i_k}^{k-1} \left(1-u_{i_k}\right)^{i-k} du_{i_k}$$

Note that for a specific values of m and $a's$, the value of the above probability can be easily computed. For $a_{ki} = 1$ for $k = i$, $i = 1,2,...,m$ and $a_{ki} = 0$, otherwise; i.e. **Ranking is perfect (no ranking error)**, $\pi(i_1, i_2,..., i_m) =$

$$\pi_0(i_1, i_2,..., i_m) = \frac{m!}{\prod_{k=1}^{m} \sum_{j=1}^{k} i_j}.$$

4. Test Statistics

In this section, some simple nonparametric tests will be introduced, these tests can be used to test if the ranking is perfect or not for one cycle MERSS. Under perfect ranking, the value of $\pi(i_1, i_2,..., i_m)$ gets larger as the distance between $(i_1, i_2,..., i_m)$ and $(1,2,\ldots,m)$ gets smaller and vise versa. That means, a suitable test statistic may be based on the distance between these two vectors. The smaller is the distance, the stronger is the evidence that H_0 is true and vise versa. The following three tests are analogue of the tests that were used by Li and Balakrishnan (2008) for one cycle RSS:

a. N_m : It is the number of inversions in the vector $(i_1, i_2,..., i_m)$, where an inversion is the presence of a pair (i_r, i_s) with $\Delta_{rs} = (r-s)(i_r - i_s) < 0$, (i.e. there is a conflict between the order and the value of the order statistics). N_m can be written as:

$$N_m = \sum_{r=1}^{m} \sum_{s=1}^{r-1} I(i_r < i_s),$$

where, $I(i_r < i_s) = 1$ if $i_r < i_s$ and zero otherwise. Clearly, the possible values of N_m are $0,1,2,..., m(m-1)/2$. The largest value of N_m, occurs when, $N_m = \sum_{r=1}^{m}(r-1) = \frac{m(m-1)}{2}$. H_0 is rejected if $N_m > c$, where c is obtained so that the significant level of the test is α, i.e. $P_{H_0}(N_m > c) \leq \alpha$.

b. S_m : It is the sum of square of $(i_r - r)$, $r = 1,..., m$, i.e.,

$$S_m = \sum_{r=1}^{m}(i_r - r)^2$$

It can be verified that the possible values of S_m are the even numbers:

$$0,2,4,6,\ldots,\frac{1}{3}m(m^2-1)$$

The largest value can be obtained as:

$$Max(S_m) = 2\sum_{r=1}^{m} r^2 - 2Max(\sum_{r=1}^{m} r i_r) = 2\sum_{r=1}^{m} r^2 - 2(\sum_{r=1}^{m} r(m-r+1) = \frac{1}{3}m(m^2-1).$$

We reject H_0 if $S_m > c$, where c is obtained using $P_{H_0}(S_m > c) \leq \alpha$.

c. A_m : It is the sum of the absolute value of the difference between i_r and r where $r = 1,..., m$ i.e.,

$$A_m = \sum_{r=1}^{m} |i_r - r|$$

The values of A_m are the set of the even numbers:

$$0,2,4,\ldots,[m^2/2].$$

Reject H_0 if $A_m > c$, where c is obtained using $P_{H_0}(A_m > c) \leq \alpha$.

We can use the above test statistics to identify the rejection region for testing the hypothesis of perfect ranking. Also, the tests can be compared via their powers. Table (1) contains some specific H_1 that can be of interest. The probabilities of ordering under specific H_1 are given in Table (2).

Table (1): Specific H_1

Case	m=2	m=3	m=4	m=5
(1)	$H_1: a_{11}=1,$ $a_{12}=a_{22}=\frac{1}{2}$	$H_1: a_{11}=1,$ $a_{12}=a_{22}=\frac{1}{2},$ $a_{13}=a_{23}=a_{33}=\frac{1}{3}$	$H_1: a_{11}=1, a_{12}=a_{22}=\frac{1}{2},$ $a_{13}=a_{23}=a_{33}=\frac{1}{3}$	$H_1: a_{11}=1, a_{12}=a_{22}=\frac{1}{2},$ $a_{13}=a_{23}=a_{33}=\frac{1}{3}$ $a_{14}=a_{24}=a_{34}=a_{44}=\frac{1}{4}$ $a_{15}=a_{25}=a_{35}=a_{45}=$ $a_{55}=\frac{1}{5}$
(2)	$H_1: a_{11}=1,$ $a_{12}=a_{22}=\frac{1}{2}$	$H_1: a_{11}=1,$ $a_{12}=a_{22}=\frac{1}{2},$ $a_{13}=a_{23}=a_{33}=\frac{1}{3}$	$H_1: a_{11}=1, a_{12}=\frac{4}{5}, a_{22}=\frac{1}{5},$ $a_{13}=\frac{8}{9}, a_{23}=\frac{1}{10}, a_{33}=\frac{1}{10}$ $a_{14}=\frac{7}{12}, a_{24}=\frac{3}{12},$ $a_{34}=\frac{1}{12}, a_{44}=\frac{1}{12}$	$H_1: a_{11}=1, a_{12}=\frac{4}{5}, a_{22}=\frac{1}{5},$ $a_{13}=\frac{9}{10}, a_{23}=0, a_{33}=\frac{1}{10}$ $a_{14}=\frac{7}{12}, a_{24}=\frac{3}{12},$ $a_{34}=0, a_{44}=\frac{2}{12}$ $a_{15}=\frac{15}{24}, a_{25}=\frac{5}{24},$ $a_{35}=0, a_{45}=\frac{1}{24},$ $a_{55}=\frac{3}{24}$
(3)	$H_1: a_{11}=1, a_{12}=\frac{1}{10},$ $a_{22}=\frac{9}{10}$	$H_1: a_{11}=1, a_{12}=\frac{1}{5},$ $a_{22}=\frac{4}{5},$ $a_{13}=\frac{1}{10}, a_{23}=0,$ $a_{33}=\frac{9}{10}$	$H_1: a_{11}=1, a_{12}=\frac{1}{5}, a_{22}=\frac{4}{5},$ $a_{13}=\frac{1}{10}, a_{23}=0, a_{33}=\frac{9}{10}$ $a_{14}=\frac{1}{12}, a_{24}=0,$ $a_{34}=\frac{1}{12}, a_{44}=\frac{10}{12}$	$H_1: a_{11}=1, a_{12}=\frac{1}{5},$ $a_{22}=\frac{4}{5},$ $a_{13}=\frac{1}{10}, a_{23}=0, a_{33}=\frac{9}{10}$ $a_{14}=\frac{1}{12}, a_{24}=0,$ $a_{34}=\frac{1}{12}, a_{44}=\frac{10}{12}$ $a_{15}=\frac{3}{24}, a_{25}=\frac{1}{24},$ $a_{35}=0, a_{45}=\frac{5}{24},$ $a_{55}=\frac{15}{24}$

Table (2): Values of $\pi_1(i_1, i_2, ..., i_m)$ under a specific H_1.

m	Case (1)	Case (2)	Case (3)
2			
(1,2)	0.5	0.444444	0.633333
(2,1)	0.5	0.555556	0.366667
3			
(1,2,3)	0.166667	0.092	0.292
(1,3,2)	0.166667	0.119	0.207
(2,1,3)	0.166667	0.114	0.201
(2,3,1)	0.166667	0.212	0.117
(3,1,2)	0.166667	0.189	0.101
(3,2,1)	0.166667	0.274	0.082
4			
(1,2,3,4)	0.041666667	0.019650794	0.106732
(1,3,2,4)	0.041666667	0.023976190	0.077286
(1,2,4,3)	0.041666667	0.018952381	0.094825
(1,4,2,3)	0.041666667	0.022658730	0.059738
(1,4,3,2)	0.041666667	0.028357143	0.049071
(1,3,4,2)	0.041666667	0.028880952	0.056512
(2,1,3,4)	0.041666667	0.022492063	0.074948
(2,3,4,1)	0.041666667	0.056976190	0.029286
(2,4,3,1)	0.041666667	0.055785714	0.025679
(2,4,1,3)	0.041666667	0.031896825	0.034845
(2,1,4,3)	0.041666667	0.021698413	0.066365
(2,3,1,4)	0.041666667	0.033880952	0.044726
(3,2,1,4)	0.041666667	0.040817460	0.033325
(3,1,2,4)	0.041666667	0.032976190	0.041000
(3,2,4,1)	0.041666667	0.067563492	0.020984
(3,4,1,2)	0.041666667	0.058857143	0.016095
(3,4,2,1)	0.041666667	0.083269841	0.014349
(3,1,4,2)	0.041666667	0.038992063	0.028806
(4,1,2,3)	0.041666667	0.030738095	0.030706
(4,2,3,1)	0.041666667	0.065357143	0.017310
(4,2,1,3)	0.041666667	0.037912698	0.024841

(4,1,3,2)	0.041666667	0.037785714	0.024131
(4,3,1,2)	0.041666667	0.058174603	0.015099
(4,3,2,1)	0.041666667	0.082349206	0.013341

5. Null Distribution and Critical values of the Test statistics

Table (3) contains the null distribution for each of the three test statistics. Table (4) contains the critical values (CV) of the three tests for nominal levels near 0.05 and 0.1 and the corresponding exact levels. From Tables (3) and (4), it can be seen that when m has small values the nominal levels 0.05 and 0.1 cannot be achieved exactly so approximation values are given, for example: $m = 5, \alpha = 0.05$, we reject H_0 when $S_5 \geq 28$; with $P(S_5 \geq 28 \mid H_0) = .054198$.

The next three tables give the distributions of the test statistics N_m, S_m & A_m when $m = 2,3,4,5$, for Case (1), Case (2) and Case (3), respectively

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Table (3): Null distribution of the test statistics

N_2	Probability	N_3	Probability	N_4	Probability	N_5	Probability
0	0.666666667	0	0.333333333	0	0.133333333	0	0.044444
1	0.333333333	1	0.416666667	1	0.280952381	1	0.134055
		2	0.183333333	2	0.274047619	2	0.203903
		3	0.066666667	3	0.184047619	3	0.216006
				4	0.083571429	4	0.171984
				5	0.03452381	5	0.114552
				6	0.00952381	6	0.069022
						7	0.03085
						8	0.009319
						9	0.004807
						10	0.001058
A_2	Probability	A_3	Probability	A_4	Probability	A_5	Probability
0	0.666666667	0	0.333333333	0	0.133333333	0	0.044444
2	0.333333333	2	0.416666667	2	0.280952381	2	0.134055
		4	0.25	4	0.360714286	4	0.260570
				6	0.177777778	6	0.279339
				8	0.047222222	8	0.183460
						10	0.066313
						12	0.031818

S_2	Probability	S_3	Probability	S_4	Probability	S_5	Probability
0	0.666666667	0	0.333333333	0	0.133333333	0	0.0444444
2	0.333333333	2	0.416666667	2	0.280952381	2	0.1340555
		6	0.183333333	4	0.057142857	4	0.0695531
		8	0.066666667	6	0.216904762	6	0.134351
				8	0.086666667	8	0.109913
				10	0.053571429	10	0.079076
				12	0.043809524	12	0.048985
				14	0.069285714	14	0.097601
				16	0.026984127	16	0.043034
				18	0.021825397	18	0.066790
				20	0.009523810	20	0.031851
						22	0.038721
						24	0.019374
						26	0.028053
						28	0.009848
						30	0.012280
						32	0.012743
						34	0.009312
						36	0.005289
						38	0.003668
						40	0.001058

Table (4): Critical values (CV) of the tests for nominal levels near 0.05 and 0.1 and the corresponding exact levels.

m	N_m		A_m		S_m	
	CV	Exact level	CV	Exact level	CV	Exact level
3	3	0.06666667	******	******	8	0.066666667
4	5	0.04404762	8	0.047222222	16	0.058337624
	4	0.127619	*****	*****	14	0.12817889
5	7	0.046034	12	0.031818	28	0.05419800
	6	0.115056	10	0.098131	26	0.08225100

Table (5): Distribution of test statistics when m=2,3,4,5 for Case 1, Case 2, and Case 3,

Table (5 a): Case (1)

N_2	Probability	N_3	Probability	N_4	Probability	N_5	Probability
0	0.5	0	0.166667	0	0.041666667	0	0.00833333
1	0.5	1	0.333333	1	0.125	1	0.03333332
		2	0.333333	2	0.2083333	2	0.07499997
		3	0.166667	3	0.25	3	0.12499995
				4	0.2083333	4	0.16666666
				5	0.125	5	0.18333326
				6	0.041666667	6	0.18333326
						7	0.13333328
						8	0.04999998
						9	0.03333332
						10	0.00833333
A_2	Probability	A_3	Probability	A_4	Probability	A_5	Probability
0	0.5	0	0.166667	0	0.041666667	0	0.00833333
2	0.5	2	0.333333	2	0.125	2	0.03333332
		4	0.5	4	0.291666667	4	0.09999996
				6	0.375	6	0.19999992
				8	0.166666667	8	0.29166655
						10	0.19999992
						12	0.16666666
S_2	probability	S_3	Probability	S_4	Probability	S_5	Probability
0	0.5	0	0.166667	0	0.041666667	0	0.00833333
2	0.5	2	0.333333	2	0.125	2	0.03333332
		6	0.333333	4	0.041666667	4	0.02499999
		8	0.166667	6	0.166666667	6	0.04999998
				8	0.083333333	8	0.05833331
				10	0.083333333	10	0.04999998
				12	0.083333333	12	0.03333332
				14	0.166666667	14	0.08333333
				16	0.041666667	16	0.04999998
				18	0.125	18	0.08333333
				20	0.041666667	20	0.04999998

						22	0.08333333
						24	0.04999998
						26	0.08333333
						28	0.03333332
						30	0.04999998
						32	0.05833331
						34	0.04999998
						36	0.03333332
						38	0.02499999
						40	0.00833333

Table (5 b): Case (2)

N_2	Probability	N_3	Probability	N_4	Probability	N_5	Probability
0	0.444444	0	0.092	0	0.019650794	0	0.003203470
1	0.555556	1	0.233	1	0.065420634	1	0.013613276
		2	0.401	2	0.140095238	2	0.034756250
		3	0.274	3	0.227777778	3	0.068741217
				4	0.257904761	4	0.115489877
				5	0.206801588	5	0.161264265
				6	0.082349207	6	0.209574118
						7	0.197087599
						8	0.098610745
						9	0.075458376
						10	0.022200807
A_2	Probability	A_3	Probability	A_4	Probability	A_5	Probability
0	0.444444	0	0.092	0	0.019650794	0	0.003203470
2	0.555556	2	0.233	2	0.065420634	2	0.013613276
		4	0.675	4	0.209269844	4	0.048037054
				6	0.423007934	6	0.123861753
				8	0.282650793	8	0.272357044
						10	0.246523018
						12	0.292404385
S_2	probability		S_3 Probability	S_4	Probability	S_5	Probability
0	0.444444	0	0.092	0	0.019650794	0	0.00320347

2	0.555556	2	0.233	2	0.065420634	2	0.013613276
		6	0.401	4	0.021698413	4	0.010672655
		8	0.274	6	0.118396824	6	0.024083595
				8	0.069174603	8	0.029723728
				10	0.070888889	10	0.027206939
				12	0.087714286	12	0.020752933
				14	0.199047619	14	0.055557438
				16	0.058857143	16	0.035040343
				18	0.206801588	18	0.066333317
				20	0.082349206	20	0.046372173
						22	0.080377147
						24	0.051696517
						26	0.099673619
						28	0.046563539
						30	0.067696353
						32	0.086524132
						34	0.091739591
						36	0.061387839
						38	0.059580589
						40	0.022200807

Table (5 c): Case (3)

N_2	Probability	N_3	Probability	N_4	Probability	N_5	Probability
0	0.633333	0	0.292	0	0.106732	0	0.026285858
1	0.366667	1	0.408	1	0.247059	1	0.087739659
		2	0.218	2	0.268341	2	0.151667855
		3	0.082	3	0.206039	3	0.187595904
				4	0.111730	4	0.183827422
				5	0.046758	5	0.153185037
				6	0.013341	6	0.119570089
						7	0.059703678
						8	0.016938802
						9	0.010893424
						10	0.002592216
A_2	Probability	A_3	Probability	A_4	Probability	A_5	Probability
0	0.633333	0	0.292	0	0.106732	0	0.026285858

2	0.366667	2	0.408	2	0.247059	2	0.087739659
		4	0.3	4	0.350737	4	0.194895689
				6	0.236588	6	0.262033283
				8	0.058884	8	0.256658865
						10	0.114453997
						12	0.057932599
S_2	probability		S_3 Probability	S_4	Probability	S_5	Probability
0	0.633333	0	0.292	0	0.106732	0	0.026285858
2	0.366667	2	0.408	2	0.247059	2	0.087739659
		6	0.218	4	0.066365	4	0.054640912
		8	0.082	6	0.201976	6	0.097026943
				8	0.082396	8	0.097635774
				10	0.063651	10	0.069477963
				12	0.059992	12	0.044852599
				14	0.095635	14	0.093504377
				16	0.016095	16	0.046650688
				18	0.046758	18	0.072387525
				20	0.013341	20	0.045627536
						22	0.068256147
						24	0.032497657
						26	0.054459095
						28	0.020317654
						30	0.023183516
						32	0.023186577
						34	0.020944199
						36	0.010670674
						38	0.008062388
						40	0.002592216

Power Comparison

Suppose that $m = 5$ and $\alpha = 0.1$, then the rejection region and the approximate power of each of the three tests is given in Table(6). It can be seen from the table that the best test is N_5 followed by A_5.

Table (6): Power for Case 1, Case2 and Case 3, respectively,.

Case (1)

	N_5	A_5	S_5
Rejection region	{6,7,8,9,10}	{10,12}	{26,28,30,32,34,36,38,40}
Approximate power	0.40833317	0.36666658	0.34166656

Case (2)

	N_5	A_5	S_5
Rejection region	{6,7,8,9,10}	{10,12}	{26,28,30,32,34,36,38,40}
Approximate power	0.602931645	0.538927403	0.535366469

Case (3)

	N_5	A_5	S_5
Rejection region	{6,7,8,9,10}	{10,12}	{26,28,30,32,34,36,38,40}
Approximate power	0.209698209	0.172386596	0.163416319

6. Application: Trees Data

In this section, data of heights and diameter, (*X*, *Y*), of 1083 trees will be used. The data was collected by Pordan (1968). Figure (1) is a scatter plot of the height versus diameter. The correlation coefficient between the two variables is $\rho = 0.721$.

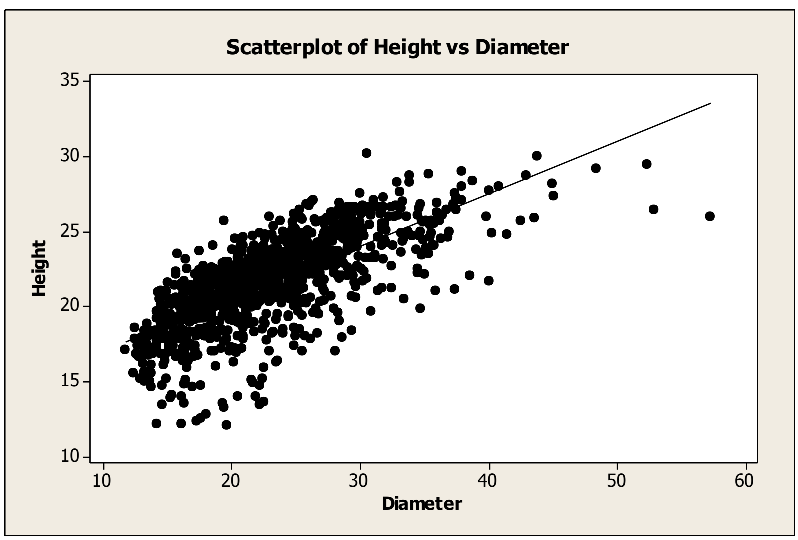

Figure (1)

Different MERSS samples are chosen from this data. Assume that (*X*, *Y*) is a bivariate data and suppose that the variable *Y* is difficult to measure or to order by judgment, but the variable X, which is highly correlated with *Y*, is easier to measure or to order by judgment.

Choose SRSs of size 1,2,…,m, respectively.

1) Identify by judgment the maximum of each set with respect to the variable *X*.
2) Measure accurately the selected judgment identified units for both variables.

This gives us a MERSS with concomitant variable.

Our hypotheses are: H_0 : Ranking is perfect(no ranking error), H_1 : there is some ranking error(ranking is imperfect).

10000 MERSSs each of size m=5, were chosen randomly as above, and the three tests for each sample are computed.

Table (7): Summary of a simulation to find the average p-value, and the power of the three test statistics using MERSS from a population of 1083 trees.

Test	Average p-value	Number of rejection	Power of the test
N_5	0.507349015	856	0.0856
A_5	0.485116479	540	0.054
S_5	0.408224868	952	0.0952

The test with smaller average p-value is S_5; it is the most sensitive test. All three tests suggest the acceptance of the hypothesis of no error in ranking.

7. Concluding Remarks

In this paper, the formula for $P(Y_{[i_1:i_1]} \le Y_{[i_2:i_2]} \le \ldots \le Y_{[i_{m1}:i_m]})$ under perfect ranking is derived; this formula is distribution free. There is no close form for $P(Y_{[i_1:i_1]} \le Y_{[i_2:i_2]} \le \ldots \le Y_{[i_{m1}:i_m]})$ under imperfect ranking, but for specific values of m and a's, this probability can be calculated easily. Using these two probabilities, the hypothesis of perfect ranking is tested versus that of imperfect ranking. Some simple non-parametric tests were investigated. For the three test statistics, the exact null distributions are obtained. Also, under error in ranking, the exact power functions are computed for some special cases. All proposed test statistics depend on the distance between $(i_1, i_2, \ldots, i_m)$ and $(1,2,\ldots,m)$; the smaller is the distance, the stronger is the evidence that H_0 is true and vice versa.

Acknowledgement

The authors are very thankful to the editor, associate editor and the referee for their comments and suggestions.

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