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Exact Moments of Record Values from Burr Distribution with Applications

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Abstract: In this paper, the exact moments of record values are obtained for two parameter Burr type XII distribution. Exact expressions for single and product moments of record statistics are also derived. The means, variances and covariances of the record statistics are computed for various values of the shape parameters. These values are then used to compute the coefficients of the best linear unbiased estimators of the location and scale parameters. The variances of these estimators are also presented. The predictors of the future record statistics are also discussed.

Keywords: Burr type XII distribution; Record statistics; Moments of record statistics; Location and scale parameters; Best linear unbiased estimators; Best linear unbiased Predictors.

1. INTRODUCTION

Extreme values frequently occurs in the lifetime data analysis, e.g. when the components are connected in series system, the system break down when the first component fails and in case of parallel connection, the system stops working when the last component fails. Moreover, in case of large data, instead of collecting the data for each observation, one may be interested in recording the record values (extremes). The choice of taking record values can be justified as in many situations the data are available in form of the extremes, i.e. only those data are recorded which are higher than previous highest value. Record values are first defined by Chandler (1952) as a model of successive extremes in a sequence of identical and independent random variables. Record values are widely used in extreme weather conditions, sports, economics and in many real life problems.

Let $X_1, X_2, \ldots$ be a sequence of independently and identically distributed (*iid*) continuous random variables (*rv*) with the cumulative distribution function (*cdf*) $F(x)$ and probability density function (*pdf*) $f(x)$ over the support $(-\infty, \infty)$. Denote the upper record times by $u(1) = 1$ and for $r > 1$

$$u(r) = \min\{k > u(r-1) : X_k > X_{u(r-1)}\}.$$

The record value sequence is then defined by $X_{u(1)}, X_{u(2)}, \ldots$.

The *pdf* of $X_{u(r)}$ is given by (Ahsanullah, 1995)

$$f_{X_{u(r)}}(x) = \frac{1}{(r-1)!}[-\log \bar{F}(x)]^{r-1} f(x). \tag{1.1}$$

The joint *pdf* of $X_{u(r)}$ and $X_{u(s)}$, $1 \le r < s$, is given by

$$f_{X_{u(r)}, X_{u(s)}}(x, y) = \frac{1}{(r-1)!(s-r-1)!}[-\log \bar{F}(x)]^{r-1}$$

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$$\times [-\log \bar{F}(y) + \log \bar{F}(x)]^{s-r-1} \frac{f(x)}{\bar{F}(x)} f(y), \qquad -\infty < x < y < \infty \tag{1.2}$$

where $\bar{F}(x) = 1 - F(x)$.

A random variable X is said to have Burr Type XII distribution if the *pdf* of X is of the form

$$f(x) = \mu\, p\, x^{p-1} [1 + x^p]^{-(\mu+1)} \quad ; \quad x > 0 \quad , \quad \mu,\ p > 0 \tag{1.3}$$

$$= 0 \qquad \text{otherwise,}$$

with the *cdf* is given as

$$F(x) = 1 - [1 + x^p]^{-\mu}. \tag{1.4}$$

Therefore for the Burr distribution, we have

$$f(x) = \frac{\mu\, p\, x^{p-1}}{[1 + x^p]} \bar{F}(x). \tag{1.5}$$

Burr type XII distribution was introduced by Burr (1942). The Burr XII distribution is widely used in reliability analysis as a more fexible alternative to Weibull distribution (e.g. Wingo, 1993; Zimmer *et al.*, 1998). Rodriguez (1977) and Tadikamalla (1980) has widely studied the Burr distribution. Tadikamalla (1980) has established its relationship with some other distributions. Sufficient literatures are available regarding the estimation of parameters of Burr distribution using ordered random variables i.e. in case of progressive censoring one may refer to Wingo (1993), Ali Mousa and Zaheen (2002), Wu (2003), Soliman(2005), Wu and Yu (2005), Wu *et al.* (2007) and Soliman *et al.* (2011). Bayesian inference based on record values was considered by Wang and Shi (2010) and Nadar and Papadopoulos (2011). Asgharzadeh and Abdi (2012) obtained the confidence intervals for the Parameters of the Burr Type XII distribution Based on Records. Pawlas and Szynal (1999), Saran and Pushkarna (2000) obtained the recurrence relations for single and product moments of k^{th} record values. In this paper, we have obtained the explicit expression for the single and product moment of Burr type XII distribution based on record values. This paper is divided in to three sections. In section 2, we have obtained the ratio and inverse moments of record statistics from Burr XII distribution. In section 3, using the result obtained in section 2, we have obtained the best linear unbiased estimator (BLUE) for location and scale parameter of Burr XII distribution for known shape parameter. Further best linear unbiased predictor (BLUP) is also calculated by utilising the results obtained in section 2.

2. COMPUTATION OF MOMENTS

In this section, we have derived the exact expressions for ratio and inverse moments of record statistics from Burr XII distribution. Further, these results are used to evaluate the means, variances and covariances of record statistics from Burr XII distribution $(p = 2,\ \mu = 2.5 : 0.5 : 6)$ for the first five records and these are presented in Table 1 and in Table 2 respectively. Moreover, these values are used to obtain the coefficients required for the best linear unbiased estimators of the location and scale parameters of Burr XII distribution based on some observed record statistics in section 3. The coefficients of the estimators are presented for various choices of observed records in Tables 3 and 4 respectively. Variances and covariances of the estimators have also been computed and presented in Table 5.

Some auxiliary results: Here some results are given which are used to obtain the moments of record statistics from Burr XII distribution.

(i) $$\frac{\partial^r \mathrm{B}(a,b)}{\partial b^r} = \sum_{k=0}^{r-1} \binom{r-1}{k} [\psi^{(r-k-1)}(b) - \psi^{(r-k-1)}(a+b)] \frac{\partial^k \mathrm{B}(a,b)}{\partial b^k},$$

where $\mathrm{B}(a,b), a,b > 0,$ is the beta function, $\psi^{(k)}(x)$ is the k^{th} derivative of $\psi(x) = \frac{d \log \Gamma(x)}{dx} = \frac{\Gamma'(x)}{\Gamma(x)}, x \neq 0, -1, -2, \ldots$

which is a diagamma function and $\Gamma(.)$ is the gamma function,.

(ii) Prudnikov *et al.* (1986, 2.6.3.2. p. 488, Vol. I)

$$\int_0^1 x^{\alpha-1}[\log x]^n \, dx = \frac{(-1)^n n!}{\alpha^{n+1}}$$

(iii) Prudnikov *et al.* (1986, 2.7.3.3. p. 488, Vol. I)

$$\int_0^a x^{\alpha-1}(a^\delta - x^\delta)^{\beta-1}[\log x]^n \, dx = \frac{a^{\delta(\beta-1)}}{\delta}\frac{\partial^n}{\partial\alpha^n}\left[a^\alpha \mathrm{B}(\beta,\frac{\alpha}{\delta})\right] \quad , \quad a,\delta,\alpha,\beta > 0.$$

Theorem 2.1: Single moments of record statistics from Burr XII distribution is given by

$$\alpha_r^{j-p} = E[X_{u(r)}^{j-p}] = \frac{(-1)^{r-1}\mu^r}{(r-1)!}\frac{\partial^{r-1}}{\partial\nu^{r-1}}\mathrm{B}\left(\frac{j}{p},\nu\right), \tag{2.1}$$

where $\nu = \mu - \frac{j}{p} + 1 > 0$, $\frac{j}{p} > 0$.

Proof: We have,

$$E[X_{u(r)}^{j-p}] = \frac{1}{(r-1)!}\int_0^\infty x^{j-p}[-\log\overline{F}(x)]^{r-1} f(x)\, dx$$

$$= \frac{\mu p}{(r-1)!}\int_0^\infty x^{j-p}[-\log(1+x^p)^{-\mu}]^{r-1}\frac{x^{p-1}}{(1+x^p)^{\mu+1}}\, dx$$

Set $t = \frac{1}{[1+x^p]}$ to get,

$$E[X_{u(r)}^{j-p}] = \frac{(-1)^{r-1}\mu^r}{(r-1)!}\int_0^1 [\log t]^{r-1} t^{\mu-\frac{j}{p}}(1-t)^{\frac{j}{p}-1}\, dt$$

Now the result is proved in view of auxiliary result (iii). Using auxiliary result (i) recursively, we can obtain the moments for any value of r.

Theorem 2.2: The Product moments of record values from Burr XII distribution

$$\alpha_{r,s}^{j\,l-p} = E[X_{u(r)}^{j}X_{u(s)}^{l-p}] = \frac{\mu^s(-1)^{r-1}}{(r-1)!}\sum_{i=0}^{\infty}\frac{\Gamma\left(1-\frac{l}{p}+i\right)}{\Gamma\left(1-\frac{l}{p}\right)i!}\frac{1}{\left(\mu-\frac{l}{p}+i+1\right)^{s-r}}\frac{\partial^{r-1}}{\partial\eta^{r-1}}\mathrm{B}\left(\frac{j}{p}+1,\eta\right),$$

$$1-\frac{l}{p} \neq 0,-1,-2,\ldots \tag{2.2}$$

and

$$\alpha_{r,s}^{j\,l-p} = \frac{\mu^s(-1)^{r-1}}{(r-1)!}\sum_{i=0}^{\frac{l}{p}-1}\frac{(-1)^i\binom{\frac{l}{p}-1}{i}}{\left(\mu-\frac{l}{p}+i+1\right)^{s-r}}\frac{\partial^{r-1}}{\partial\eta^{r-1}}\mathrm{B}\left(\frac{j}{p}+1,\eta\right), \qquad \frac{l}{p}-1\in \mathrm{N}, \tag{2.3}$$

where $\eta = \mu - \frac{j+l}{p} + i + 1 > 0$ and $\frac{j}{p} + 1 > 0$.

Proof: We have,

$$\alpha_{r,s}^{j\,l-p} = \frac{\mu\, p}{(r-1)!} \int_0^\infty x^j [-\log(1+x^p)^{-\mu}]^{r-1} \frac{x^{p-1}}{[1+x^p]} I(x)\, dx \tag{2.4}$$

where

$$I(x) = \frac{\mu\, p}{(s-r-1)!} \int_x^\infty y^{l-p} [-\log(1+y^p)^{-\mu} + \log(1+x^p)^{-\mu}]^{s-r-1} \frac{y^{p-1}}{[1+y^p]^{\mu+1}}\, dy$$

Set $t = \dfrac{(1+x^p)}{(1+y^p)}$, to get

$$I(x) = \frac{\mu\,(-\mu)^{s-r-1}}{(s-r-1)!\,((1+x^p)^{\mu-\frac{l}{p}+1}} \int_0^1 \left[1-\frac{t}{(1+(x)^p)}\right]^{\frac{l}{p}-1} [\log t]^{s-r-1}\, t^{\mu-\frac{l}{p}}\, dt$$

Now using the Maclaurin series expansion, we have

$$\left[1-\frac{t}{(1+x^p)}\right]^{\frac{l}{p}-1} = \sum_{i=0}^{\infty} \frac{\Gamma\left(1-\frac{l}{p}+i\right)}{\Gamma\left(1-\frac{l}{p}\right)} \frac{\left(\frac{t}{(1+x^p)}\right)^i}{i!}$$

Hence using the result (ii), we get

$$I(x) = \mu^{s-r} \sum_{i=0}^{\infty} \frac{\Gamma\left(1-\frac{l}{p}+i\right)}{\Gamma\left(1-\frac{l}{p}\right) i!} \frac{1}{(1+x^p)^{\mu-\frac{l}{p}+i+1} \left(\mu+i-\frac{l}{p}+1\right)^{s-r}}$$

Putting the value of $I(x)$ in (2.4) and setting $t = \dfrac{1}{(1+x^p)}$, we get

$$\alpha_{r,s}^{j\,l-p} = \frac{\mu^s (-1)^{r-1}}{(r-1)!} \sum_{i=0}^{\infty} \frac{\Gamma\left(1-\frac{l}{p}+i\right)}{\Gamma\left(1-\frac{l}{p}\right) i!} \frac{1}{\left(\mu+i-\frac{l}{p}+1\right)^{s-r}} \int_0^1 [\log t]^{r-1}\, t^{\mu-\frac{j+l}{p}+i} (1-t)^{\frac{j}{p}} dt,$$

Now using the auxiliary result (iii), the Theorem is proved. Proceeding in the similar way and expanding $\left[1-\dfrac{t}{(1+x^p)}\right]^{\frac{l}{p}-1}$ binomially, (2.3) can be established.

Remark 2.1: Theorem 2.2 reduces to Theorem 2.1, if we put $j = j-p$ and $l = p$ in (2.3).

Remark 2.2: The results obtained in (2.1), (2.2) and in (2.3) can be utilized for obtaining the moments of Lomax distribution $(p=1)$ and log logistic distribution $(\mu=1)$ as they are the special cases of Burr XII distribution.

Table 1: Mean of record statistic from Burr distribution $(p=2)$

r	$\mu=2.5$	$\mu=3$	$\mu=3.5$	$\mu=4$	$\mu=4.5$	$\mu=5$	$\mu=5.5$	$\mu=6$
1	0.6667	0.589	0.5333	0.4909	0.4571	0.4295	0.4063	0.3866
2	1.134	0.9772	0.87	0.7912	0.7302	0.6813	0.641	0.607
3	1.6198	1.358	1.1873	1.0658	0.9743	0.9022	0.8438	0.7952
4	2.1753	1.7703	1.5178	1.3439	1.2159	1.1171	1.0383	0.9735
5	2.8362	2.2362	1.8777	1.6384	1.4664	1.3362	1.2337	1.1507

Table 2: Covariance of record statistic from Burr distribution $(p=2)$

r s	$\mu=2.5$	$\mu=3$	$\mu=3.5$	$\mu=4$	$\mu=4.5$	$\mu=5$	$\mu=5.5$	$\mu=6$
1 1	0.22221	0.1538	0.11559	0.0924	0.07676	0.06553	0.05722	0.05054
1 2	0.23016	0.1486	0.10723	0.0829	0.067226	0.056182	0.048162	0.04203
1 3	0.26328	0.1604	0.11111	0.0834	0.065947	0.054205	0.045764	0.03947
1 4	0.31342	0.1811	0.12076	0.0879	0.068112	0.055006	0.045839	0.03894
1 5	0.38060	0.2092	0.13452	0.0954	0.072409	0.057402	0.047148	0.03963
2 2	0.49184	0.2951	0.2031	0.1518	0.119908	0.09833	0.082919	0.07155
2 3	0.56694	0.3207	0.21165	0.1534	0.118266	0.095231	0.079124	0.06731
2 4	0.67761	0.3633	0.23071	0.1624	0.12245	0.09682	0.07925	0.06668
2 5	0.82485	0.4205	0.25760	0.1764	0.130235	0.101147	0.081698	0.06782
3 3	1.00585	0.5308	0.33432	0.2345	0.17614	0.139135	0.113802	0.09566
3 4	1.20644	0.6032	0.36551	0.2489	0.182749	0.141852	0.114282	0.09487
3 5	1.47152	0.6995	0.40871	0.2707	0.194686	0.14838	0.117904	0.09676
4 4	1.98407	0.9285	0.53788	0.3544	0.254187	0.193488	0.153433	0.12589
4 5	2.42421	1.0785	0.60243	0.3861	0.271104	0.202731	0.158549	0.12849
5 5	3.81607	1.5932	0.85244	0.5296	0.363071	0.26637	0.205384	0.16419

3. Application of the moments

The exact and explicit expressions for single moments of record statistics given in (2.1) allow us to evaluate the means of all record statistics. Table 1 presents the means of $X_{u(r)}$ $r=1,\ldots,5$, for Burr XII distribution. For the computation of variances and covariances, the product moments $\alpha_{r,s}^{1,1}, 1\le r<s$ were computed first and then variances and covariances are computed. For $r\ge s$ the values of $\sigma_{r,s}$ were filled in by using the symmetry of the variance–covariance matrix $(\sigma_{r,s})$. Table 2 provides the variances and covariances of record statistics for $p=2,\ \mu=2.5:0.5:6$. MATLAB has been used for computation of the moments as beta and polygamma functions are available there.

Assume $Y_1, Y_2\ \ldots$ to be an infinite sequence of iid *r.v.'s* with *pdf*

$$g(y)=\mu\, p\left(\frac{y-\theta}{\sigma}\right)^{p-1}\left[1+\left(\frac{y-\theta}{\sigma}\right)^{p}\right]^{-(\mu+1)} \quad ; \quad y>\theta\ ,\quad \mu,\ p,\sigma>0 \tag{3.1}$$

Let $Y_{u(1)} \le Y_{u(2)} \ldots \le Y_{u(s)}$ be the first s observed record values from the above sequence. Then where $X_{u(i)} = \left(\frac{Y_{u(i)} - \theta}{\sigma}\right)$, $i = 1,2,\ldots,s$ is the vector of s observed record statistics from a population with the standard Burr XII distribution *pdf* and *cdf* given in (1.3) and (1.4) respectively. Then we can write the best linear unbiased estimators (BLUE's) of θ and σ (Arnold *et al*., 1998, pp. 133–142) as

$$\hat{\theta} = a_1 Y_{u(1)} + a_2 Y_{u(2)} + \ldots + a_s Y_{u(s)} \tag{3.2}$$

$$\hat{\sigma} = b_1 Y_{u(1)} + b_2 Y_{u(2)} + \ldots + b_s Y_{u(s)} \tag{3.3}$$

Here a_i's and b_i's are the entries of the matrix $C = (A'V^{-1}A)^{-1}A'V^{-1}$ with

$A = (\underline{1} \ \ \underline{\alpha})$, $\underline{1}' = (1,\ldots,1)_{1\times s}$, $\underline{\alpha}' = (\alpha_1,\ldots,\alpha_s)_{1\times s}$, where $\underline{\alpha}$ is the mean of the first s record values and V^{-1} is the inverse of the covariance matrix $V = (\sigma_{r,s})_{s\times s}$. Variances and covariance of these estimators are given by

$$Var(\hat{\theta}) = d_{11}\sigma^2, \ Var(\hat{\sigma}) = d_{22}\sigma^2 \text{ and } Covar(\hat{\theta},\hat{\sigma}) = d_{12}\sigma^2, \tag{3.4}$$

where

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}\sigma^2 = (A'V^{-1}A)^{-1}.$$

The necessary coefficients in (3.2) and (3.3) required for the BLUE of θ and σ based on of record statistics from the Burr XII distribution $(p = 2)$ are computed in Table 3 and in Table 4 respectively. The variances and covariances of the BLUE's are presented in Table 5. Here coefficients are computed for Burr XII distribution $(p = 2,\ \mu = 2.5 : 0.5 : 6)$. Similarly we can obtain BLUE for any other choice of the parameters μ, p.

Table 3: Coefficients for the BLUE of θ for $p = 2$

s	$\mu = 2.5$	$\mu = 3$	$\mu = 3.5$	$\mu = 4$	$\mu = 4.5$	$\mu = 5$	$\mu = 5.5$	$\mu = 6$
2	2.4267	2.5173	2.5839	2.6347	2.6737	2.7057	2.7311	1.8333
	-1.4267	-1.5173	-1.5839	-1.6347	-1.6737	-1.7057	-1.7311	-0.8333
3	1.7828	1.8039	1.822	1.8383	1.851	1.8635	1.8715	1.2156
	-0.1634	-0.0767	-0.0135	0.0326	0.0695	0.0965	0.1233	0.2992
	-0.6194	-0.7272	-0.8085	-0.8709	-0.9205	-0.96	-0.9949	-0.5148
4	1.5777	1.5648	1.5591	1.5583	1.5602	1.5632	1.5635	1.0126
	-0.1272	-0.046	0.0106	0.051	0.0792	0.1008	0.1221	0.2276
	-0.1302	-0.1013	-0.0725	-0.0484	-0.0267	-0.0078	0.0086	0.1295
	-0.3202	-0.4176	-0.4971	-0.5608	-0.6127	-0.6562	-0.6942	-0.3698
5	1.4868	2.2973	1.4293	1.4168	1.4095	1.4063	1.4009	0.9146
	-0.1128	-0.0522	0.0214	0.0587	0.0825	0.1028	0.1248	0.1884
	-0.1188	-0.1381	-0.0586	-0.036	-0.0121	0.0028	0.0158	0.1068
	-0.0801	-0.1207	-0.0693	-0.056	-0.0437	-0.0339	-0.0283	0.0756
	-0.1751	-0.4012	-0.3228	-0.3835	-0.4362	-0.478	-0.5132	-0.2853
	2.4267	2.5173	2.5839	2.6347	2.6737	2.7057	2.7311	1.8333

Table 4: Coefficients for the BLUE of σ for $p=2$

s	$\mu=2.5$	$\mu=3$	$\mu=3.5$	$\mu=4$	$\mu=4.5$	$\mu=5$	$\mu=5.5$	$\mu=6$
2	-2.14	-2.576	-2.97	-3.33	-3.6617	-3.9714	-4.2608	-4.1667
	2.14	2.576	2.97	3.33	3.6617	3.9714	4.2608	4.1667
3	-1.2443	-1.4542	-1.6467	-1.8273	-1.9915	-2.1527	-2.3005	-1.5808
	0.3829	0.3105	0.2424	0.184	0.1229	0.0796	0.0318	-0.5741
	0.8615	1.1436	1.4042	1.6433	1.8686	2.0731	2.2686	2.1549
4	-0.9782	-1.1043	-1.2223	-1.3347	-1.4429	-1.5484	-1.645	-0.8362
	0.3359	0.2656	0.2036	0.1517	0.1046	0.0709	0.0344	-0.3115
	0.2266	0.2277	0.2161	0.1965	0.1823	0.1572	0.1334	-0.2087
	0.4157	0.611	0.8026	0.9865	1.156	1.3203	1.4771	1.3564
5	-0.8671	-1.5786	-1.0252	-1.1014	-1.1766	-1.2517	-1.32	-0.5144
	0.3182	0.3335	0.1871	0.1389	0.0989	0.0672	0.0291	-0.183
	0.2126	0.2941	0.1949	0.176	0.1565	0.1371	0.1189	-0.134
	0.1222	0.2038	0.1531	0.1542	0.1508	0.1436	0.1461	-0.1051
	0.214	0.5164	0.49	0.6323	0.7705	0.9039	1.0259	0.9365
	-2.14	-2.576	-2.97	-3.33	-3.6617	-3.9714	-4.2608	-4.1667

Table 5: Variances and covariances of the BLUE of θ and σ in terms of σ^2 ($p=2$)

s	$\mu=2.5$	$\mu=3$	$\mu=3.5$	$\mu=4$	$\mu=4.5$	$\mu=5$	$\mu=5.5$	$\mu=6$
2	0.4039	0.333	0.2835	0.2468	0.2193	0.0911	0.4039	0.333
	0.92	0.8694	0.8353	0.8107	0.7933	0.6615	0.92	0.8694
	-0.5156	-0.4584	-0.417	-0.3851	-0.3607	-0.1677	-0.5156	-0.4584
3	0.2743	0.2245	0.1905	0.1652	0.1465	0.0668	0.2743	0.2245
	0.5293	0.4831	0.4523	0.4304	0.4148	0.2348	0.5293	0.4831
	-0.2906	-0.2537	-0.2283	-0.209	-0.1947	-0.0658	-0.2906	-0.2537
4	0.2305	0.187	0.1578	0.1364	0.1206	0.0586	0.2305	0.187
	0.4149	0.3672	0.3357	0.3138	0.2973	0.1241	0.4149	0.3672
	-0.2198	-0.1878	-0.1665	-0.1511	-0.1395	-0.0356	-0.2198	-0.1878
5	0.2093	0.1684	0.141	0.1215	0.1072	0.0543	0.2093	0.1684
	0.3662	0.3165	0.2835	0.2605	0.2438	0.078	0.3662	0.3165
	-0.1877	-0.1571	-0.137	-0.1229	-0.1127	-0.0216	-0.1877	-0.1571

Example. Let us consider the case where the components have failure times which follow a Burr XII distribution with $(\mu, p, \theta, \sigma) = (3, 2, 10, 4)$. Suppose that we observe the following simulated observed failure times.

10.9105 , 12.0759, 11.3485, 11.2747, 12.6274, 10.7958, 11.7932, 11.7818, 10.9605, 11.3118, 10.3771, 10.3156, 11.237, 11.8683

Therefore, we observe the record statistics from the observed data as follows:

10.9105 , 12.0759, 12.6274

Here then, for the recorded data analysis, with $s=3$, $\mu=3$ and $p=2$; $\underline{\alpha}$, and V are obtained from Table 1 and Table 2 respectively. The coefficients in (3.2) and (3.3) are presented in Table 3 and in Table 4 respectively.

Therefore, the BLUE's of θ and σ are computed to be $\hat{\theta}=9.5726$ and $\hat{\sigma}=2.3242$. The corresponding variances and covariances of $\hat{\theta}$ and $\hat{\sigma}$ (see Table 5) are computed to be

$$V(\hat{\theta})=0.2245\sigma^2=3.5920 \text{ and } V(\hat{\sigma})=0.4831\,\sigma^2=7.7296 \text{ and } Cov(\hat{\theta},\hat{\sigma})=\text{-}0.2537\sigma^2=-4.0592$$

Let us consider the true population mean $\tau=E(Y)=\theta+\alpha_1\,\sigma=12.536$. Now suppose $\tau^*=[Y_{u(1)}+Y_{u(2)}+Y_{u(3)}]/3$ is the mean of the observed records. We would have $\tau^*=11.871267$ and $S.E.(\tau^*)=1.9951$. The BLUE of τ is $\hat{\tau}=\hat{\theta}+\alpha_1\,\hat{\sigma}=10.9415$. The standard error of $\hat{\tau}$ is computed to be $S.E.(\hat{\tau})=1.2214$. Therefore, the BLUE performs better than the mean of observed records in the sense of standard error.

In the context of prediction of the future record observations, suppose we observe only the first r recorded observations $\underline{Y}=(Y_{u(1)},Y_{u(2)},\ldots,Y_{u(r)})$ and the goal is to predict $Y_{u(s)}$, where $1\le r<s$. When F belongs to a location and scale parameter family, the most well-known predictor is the best linear unbiased predictor (BLUP) (see, for example in the context of order statistics, Kaminsky and Nelson, 1975) of $Y_{u(s)}$ is given by

$$\hat{Y}_{u(s)}=(\hat{\theta}+\hat{\sigma}\alpha_s)+\underline{w}'V^{-1}(\underline{X}-\hat{\theta}.\underline{1}-\hat{\sigma}\underline{\alpha}),$$

where $\underline{\alpha}$ is the mean of the first r record values and $\underline{w}'$ is the vector of the covariances between the s^{th} future record statistic and the first r recorded observations. The mean square prediction error (MSPE) of $Y_{u(s)}$ is found to be (see, for example, Raqab, 1996)

$$MSPE[Y_{u(s)}]=[\underline{E}'V\,\underline{E}+\sigma_{n,n}-2\,\underline{E}'\,\underline{w}]\sigma^2,$$

where

$$\underline{E}'=(1.\alpha_n)(A'V^{-1}A)^{-1}A'V^{-1}+\underline{w}'V^{-1}(I-A(A'V^{-1}A)^{-1}A'V^{-1}).$$

In our data setup, we have observed three record statistics. Tables 1, 2, 3, 4 and 5 are used to compute the BLUP of the future record statistics $Y_{u(4)}$ and $Y_{u(5)}$ based on the first three observed record statistics. Their values are computed to be

$$Y_{u(4)}=13.5697 \text{ and } Y_{u(5)}=14.6322$$

and and the corresponding MSPE's are given by

$$MSPE(Y_{u(4)})=0.3016\,\sigma^2=4.8263 \qquad MSPE(Y_{u(5)})=0.3796\,\sigma^2=6.0738$$

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References

[1] M. Ahsanullah, " Records Statistics," Nova Science Publishers, New York, 1995.

[2] M. A. M. Ali Mousa and Z. F. Jaheen, "Statistical inference for the Burr model based on progressively censored data," Comp. Math. Appl., vol. 43, 2002, pp.1441-1449.

[3] B.C. Arnold, N. Balakrishnan and H.N. Nagaraja, "Records," Wiley, New York, 1998.

[4] A. Asgharzadeh and M. Abdi "Confidence intervals for the parameters of the Burr Type XII distribution based on Records," Int. Journ. Statist. and Econ., vol. 8, 2012, pp. 96-114

[5] I.W. Burr, "Cumulative frequency functions," Ann. Math. Statist., vol. 13, 1942, pp. 215–222.

[6] K. N. Chandler, "The distribution and frequency of record values," J. R. Statist. Soc., vol. B14, 1952, pp. 220-228.

[7] K.S. Kaminsky and P.I. Nelson, " Best Linear Unbiased Prediction of Order Statistics in Location and Scale Families," J. Amer. Statist. Assoc., vol. 70, 1975, pp. 145–150.

[8] M. Nadar and A.S. Papadopoulos, "Bayesian analysis for the Burr Type XII distribution based on record values," Statistica, 2011, pp. 421-435.

[9] P. Pawlas and D. Szynal, "Recurrence relations for single and product moments of k^{th} record values from Pareto, Generalized Pareto and Burr distributions," Comm. Statist. Theory Methods, vol. 28, 1999, pp. 1699-1709.

[10] A. P. Prudnikov, Y. A. Brychkov and O. I. Marichev " Integral and series," Gordon and Breach Science Publishers, vol. 1, 1986.

[11] M.Z. Raqab, "Linear predictors of future order statistics," Commun. Statist.- Theory Meth., vol. 25 (7), 1996, pp. 1553–1568.

[12] R. N. Rodriguez, "A guide to the Burr type XII distributions," Biometrika, vol. 64, 1977, pp. 129 - 134.

[13] J. Saran and N. Pushkarna, "Recurrence relations for moments of record values from Burr distribution," J. Statist. Res., vol. 34, 2000, pp. 17-23.

[14] A. A. Soliman, "Estimation of parameters of life from progressively censored data using Burr-XII model," IEEE T. Reliab., Vol. 54(1), 2005, pp. 34 – 42.

[15] A. A. Soliman, A. H. Abd Ellah, N. A. Abou-Elheggag and A. A. Modhesh, " Bayesian Inference and Prediction of Burr Type XII Distribution for Progressive First Failure Censored Sampling," Intel. Info. Manage., vol. 3, 2011, pp. 175-185.

[16] P.R. Tadikamalla, "A look at the Burr and related distributions," International Statist. Rev., vol. 48, 2005, pp. 337-344.

[17] L. Wang and Y. Shi, "Empirical Bayes inference for the Burr Model based on records," Appl. Math. Sci., vol. 4(34), 2010, pp. 1663 – 1670.

[18] D. R. Wingo, "Maximum likelihood methods for fitting the Burr type-XII distribution to multiply (progressively) censored life test data," Metrika, vol. 40, 1993, pp. 203- 210.

[19] S. J. Wu, " Estimation for the two-parameter Pareto distribution under progressive censoring with uniform removals," J. Stats. Comput. Sim., vol. 73, 2003, pp. 125-134.

[20] S. J. Wu, Y. J. Chen and , C. T. Chang, "Statistical inference based on progressively censored samples with random removals from the Burr type XII distribution," Stats. Comput. Sim., Vol. 77, 2007, pp. 19-27.

[21] J. W. Wu and H. Y. Yu, "Statistical inference about the shape parameter of the Burr type XII distribution under the failure-censored sampling plan," Appl. Math. Comput., Vol. 163, No. 1, 2005, pp. 443-482.

[22] W. J. Zimmer, J. B. Keats and F. K. Wang, "The Burr XII distribution in reliability analysis," J. Qual. Technol., vol. 30, 1998, pp. 386-394.