International Journal of Business and Statistical Analysis
ISSN (2384-4663)
Int. J. Bus. Stat. Ana. 3, No. 1 (Jan-2016)

Generalizations and Applications of L-moments

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Received 2October 2015, Revised 29 October. 2015, Accepted 30 October 2015, Published 1 January 2016

Abstract: This is a review for the book "Generalizations and applications of L-moments". The L-moments are linear functions of order statistics. In this book a generalization of L-moment is presented by giving zero weights to extreme observations, this method called trimmed L-moments (TL-moments). TL-moments have certain advantages over L-moments and method of moments. They exist whether or not the mean exists (for example the Cauchy distribution) and they are more robust to the presence of outliers.

Keywords: L-moment, order statistic, TL-moment.

1. Introduction

The subject of order statistics deals with properties and applications of ordered random variables and of functions of these variables. If the random variables $\{X_i\}$, $i = 1,2,\dots,n$ are arranged in ascending order of magnitude and then written as

$$X_{1:n} < X_{2:n} < \cdots < X_{n:n}$$

then $X_{i:n}$ is said to be the ith -order statistics in a sample of size n . In the usual random sampling theory, the unordered X_i are assumed to be statistically independent and identically distributed. Because of the inequality relations among them, the order statistics $X_{i:n}$ are necessarily dependent. Some frequently encountered functions of order statistics are the extremes $X_{1:n}$and $X_{n:n}$, the range $R = X_{n:n} - X_{1:n}$, the extreme deviate from the sample mean, $X_{n:n} - \bar{X}$, and for a random sample from a normal distribution $N(\mu,\sigma)$, the studentized range,

$$R/S_v$$

where S_v is a root mean square estimator of σ based on v degrees of freedom; see, for example, David (1981). All of these statistics have important applications. The extremes arise in the statistical study of floods and droughts, as well as in breaking strength and fatigue failure studies, the range is widely used in the field of quality control as a quick estimator of process standard deviation σ , the extreme deviate is a basic tool in procedures for detecting outliers and large values of

$$(X_{n:n} - \bar{X})/\sigma$$

suggest the presence of outliers, and when outliers are not confined to one direction, the studentized range is also useful in the detection process; see, for example, Barnett and Lewis (1994).

Sarhan and Greenberg (1962) used linear functions of order statistics in conjunction with the Gauss-Markov theorem to systematically estimate location and scale parameters in both complete and censored samples. They provided tables of the coefficients necessary for the calculations of these estimates from samples varying in size from 2 to 20 . Other applications of order statistics arise in the study of reliability systems. A system of n components is called a k-out-of-n system if it remains operational only if at least k components continue to function. For components with independent lifetime distributions, the time to failure of the system is thus the $(n-k+1)$ th -order statistic. The special cases $k = 1$ and $k = n$ correspond respectively to parallel and series systems.

A major impetus for the study of order statistics has been provided by the development of modern computers. Through their use it is feasible to make repeated examinations of the same data in many different ways. Tukey (1970) and Mosteller and Tukey (1977) have employed various informal techniques in the analysis of data. It is possible to determine quickly if the data are in accord with an assumed distribution and with an assumed model.

A plot of the ordered observations against some simple functions of their ranks, preferably on probability paper appropriate for the assumed distribution, will often prove helpful in making such determinations. The term robust statistics has many meanings, they use it in a relatively narrow sense: ...robustness signifies insensitivity to small deviations from the assumption of normality....see Huber (1981).

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Tukey (1960) points out that for a sample from $N(\mu,\sigma)$ the mean deviation has asymptotic efficiency 0:88 relative to the standard deviation in estimating σ. The situation is changed drastically if some contamination by a wider normal, for example $N(\mu,9\sigma^2)$ is present: as little as 0:008 of the wider population will render the mean deviation asymptotically superior. Nevertheless there are flaws: the efficiency of the mean is very small for a uniform parent, and for any parent a single wild observation may render X useless.

It has long been known that the midpoint,

$$(X_{1:n}+X_{n:n})/2$$

is optimal in the former case but much worse than X in the latter, and that the median is preferable in the latter case but worse in the former. Obviously, they must not expect an estimator to be good under too wide a set of circumstances.

It is standard statistical practice to summarize a probability distribution or an observed data set by some of its moments. It is also common, when fitting a parametric distribution to a data set, to estimate the parameters by equating the sample moments to those of the fitted distribution. The method of moments is not always satisfactory: sometimes it is difficult to assess exactly what information about the shape of a distribution is conveyed by its moments of third and higher order, the numerical values of sample moments, particularly when the sample is small, can be very different from those of the probability distribution from which the sample was drawn, and the estimated parameters of distributions fitted by the method of moments are often markedly less accurate than those obtainable by other estimation procedures such as the method of maximum likelihood; see, for example, Vogel and Fennessey (1993) and Kirby (1974).

Many statistical techniques are based on the use of linear combinations of order statistics but there has not been developed a unified theory covering the characterisation of probability distributions, the summarisation of observed data samples, the fitting of probability distributions to data and the testing of hypotheses about fitted distributions, until Hosking introduced L-moments in 1990, the L in L-moments emphasises the construction of L-moments from linear combinations of order statistics.

Greenwood et al. (1979) have introduced probability weighted moments, and they used them as a basis for estimating the parameters of some known distributions, for example, the Gumbel distribution. Hosking (1990) has studied an alternative approach based on quantities, which he called L-moments, which are analogous to the conventional moments but can be estimated by linear combinations of order statistics (L-statistics). L-moments have the theoretical advantages over conventional moments of being able to characterize a wider range of distributions and of being more robust to the presence of outliers of the data.

2. Chapter 2: L-moments

Hosking (1990) introduced population L-moments $\lambda_1,\lambda_2,\ldots$ as robust alternatives to classical measures of location, dispersion, skewness and kurtosis based on central moments and has studied properties of their corresponding sample L-moments $l_1,l_2,\ldots,l_n$ for samples of size n from any continuous distribution. Sample L-moments which can be expressed as linear combinations of the sample order statistics, are unbiased for the corresponding population quantities $\lambda_1,\lambda_2,\ldots,\lambda_n$, and Hosking (1990) has given expressions for their asymptotic variances and covariances. An example of a sample L-moment is Gini's mean difference scale estimate g which is twice the sample L-moment l_2 and therefore has expectation $2\lambda_2$. Nair (1936) derived the standard error of g for any continuous distributions and Lomoniki (1951) obtained in a different way a general expression for the standard error of g when sampling is from any continuous distribution.

In this chapter they derive expressions for the exact variances and covariances of sample L-moments in terms of first and second-moments of order statistics from small samples. For example, the variance of Gini's mean difference g depends only on the mean and covariance structure of the order statistics for conceptual samples of sizes 1, 2 and 3. They give examples of the use of these formulae for various distributions.

In section 2.2 they review classical moments. In sections 2.3 and 2.4 definitions and equivalent expressions for population and sample L-moments are given. In section 2.5 they derive exact results for the mean and variance-covariance structure of sample L-moments for any univariate continuous distribution. In section 2.6 they show how to derive distribution-free unbiased estimators of the variances and covariances of sample L-moments and give two examples. In section 2.7 they establish a theorem which characterises the normal distribution in terms of sample L-moments. In section 2.8 they apply these results to obtain exact variances and covariances for sample probability weighted moments.

3. Chapter 3: Generalisations of L-moments

Consider the problem of estimating the parameters of a distribution F. Classical estimation methods (e.g, the method of moments, least squares, and maximum likelihood) work well, for example, in cases where the distribution belongs to the exponential family. However, it is recognised that outliers, which arise from heavy-tailed distributions or gross errors of measurement, have undue influence on such methods; for example, $\bar{X}$ which is an unbiased estimator of the mean μ of the normal distribution based on the method of moments, least squares and maximum likelihood, is a non-robust estimator; see, for example, Ali and Luceno (1997). Therefore, if there is concern about extreme observations which having undue influence, one should use a robust method of estimation which has been developed to reduce the influence of outliers on the final estimates. In recent years, a great deal of attention has been focused on robust estimation methods; methods produce estimates that are resistant to the presence of outliers; see, for example, Barnett and Lewis (1994),

Hampel et al. (1986), Hawkins (1980) and Rousseuw and Leroy (1987). Hosking (1990) unified analysis and estimation of distributions using linear combinations of order statistics and used their ratios as new measures of skewness and kurtosis to relate L-moments to the method of moments. Royston (1992) and Vogel and Fennessey (1993) discuss the advantages of L-skewness and L-kurtosis over their product-moment counterparts. Hosking and Wallis (1995), Sillito (1951) and Sillito (1969) consider various theoretical aspects and applications of L-moments. Mudholkar and Hutson (1998) introduced LQ-moments using a "quick" measure of the location of the sampling distribution of the order statistics such as the median, the tri-mean and Gastwirth measure (which they call Gastwirth) in place of the mean. There are wide applications for L-moments in engineering, meteorology, and hydrology; see, for example; Gingras and Adamowski (1994), Guttman et al. (1993), Pearson (1993), Pilon and Adamowski (1992) and Sankarasubramanian and Srinivasan (1999).

In this chapter an alternative approach which they call trimmed L-moments (TL-moments) is introduced which gives zero weight to extreme observations. TL-moments have advantages over L-moments and the method of moments: they exist whether or not the mean exists (for example, the Cauchy distribution) and they are more robust to the presence of outliers. Trimming refers to the removal of extreme values of a sample. For example, to symmetrically trim a univariate sample size, one removes the k smallest and k largest values for some specified $k < n/2$. For univariate samples the trimmed mean, the mean of the $n - 2k$ un-trimmed sample values, is by far the most widely studied trimmed statistic.

In section 3.2 they define LQ-moments and obtain their large sample variances. In section 3.3 they introduce both population trimmed L-moments and their sample counterparts for estimating parameters from any univariate continuous distribution and also obtain their exact variances and covariances. In section 3.4 they develop the trimmed probability weighted moment method (TPWM) and elucidate its relation to TL-moments. In section 3.5 they study the TL-mean as a robust location estimator and apply the method of TL-moments to some symmetric distributions.

4. Chapter 4: Symmetric lambda istribution

Tukey (1962) introduced and discussed subsequently the very useful family of distributions defined by the single parameter quantile function

$$Q(p) = \frac{p^{\lambda} - (1-p)^{\lambda}}{\lambda}$$

Where $0 < p < 1$.

Random variables with this quantile function are said to be distributed according to a symmetric lambda distribution with parameter λ. Filliben (1969) used this distribution to approximate symmetric distributions with a wide range of tail weights to study location estimators of symmetric distributions. Joiner and Rosenblatt (1971) have given results on the sample range. Chan and Rhodin (1980) used this distribution to study robust estimation of the location parameter based on selected order statistics. Ramberg and Schmeiser (1972) have shown how this distribution can be used to approximate many of the known symmetric distributions and explored its application to Monte Carlo simulation studies. Ramberg and Schmeiser (1974) generalised $Q(p)$ to a four-parameter distribution defined by the quantile function

$$Q(p) = \lambda_1 + \frac{p^{\lambda_3} - (1-p)^{\lambda_4}}{\lambda_2}$$

where λ_1 is a location parameter, λ_1 is a scale parameter and λ_3 and λ_4 are shape parameters.

In section 4.2 they give the properties of the symmetric lambda distribution. In section 4.3 they discuss estimating the parameter l in (4.1) using the maximum likelihood method.

Also, they discuss the use of L-moments and LQ-moments for estimating the parameters. In section 4.4 they obtain the asymptotic variances of sample L-moments derived in Chapter 2. In Section 4.5 they use the symmetric lambda distribution to study the effect of the tail of the distribution on the choice of the plotting position for quantile plots.

5. Chapter 5: Control charts based on sample L-moments

The usual practice in using control charts to monitor a process is to take samples from the process at fixed-length sampling intervals and plot some sample statistics on the chart. A point outside the control limits is taken as an indication that something, called "assignable cause", has happened to change the process. Since Shewhart introduced control charts in 1924, they have found widespread application in improving the quality of manufacturing processes. Another popular control procedure is the cumulative sum (CUSUM) control chart which was introduced by Page (1954). There has also been a renewed interest in the exponentially weighted moving average (EWMA) control charts, introduced by

Roberts (1959) who called it a geometric moving average chart. It is known that Shewhart-type charts are relatively inefficient in detecting small changes in the process parameters; see, for example, Hunter (1986) and Montgomery (1996). On the other hand, EWMA charts have been shown to be more efficient than Shewhart-type charts in detecting small shifts in the process mean; see, for example, Ng and Case (1989), Crowder (1989), Lucas and Saccucci (1990), Amin and Searcy (1991) and Wetherill and Brown (1991). In fact, the EWMA control chart has become popular for monitoring the process mean; see Hunter (1986) for a good discussion. More recently, EWMA charts have been developed for monitoring process variability; see, for example, Macgregor and Harris (1993), Amin and Wollf (1995) and Gan (1995).

Like the Shewhart control chart, an EWMA control chart is easy to implement and interpret. It is based on the statistics

$$Z_i = \lambda X_i + (1-\lambda)Z_{i-1}$$

where X_i is the current observation, Z_0 is a starting value, such as the overall sample mean, and $0 < \lambda \leq 1$ is a constant that determines the "depth of memory" of the EWMA: The value $\lambda = 1$ gives the classical charts, such as the $\bar{X}$ chart. While the choice of λ can be left to the judgement of the quality control analyst. Experience with econometric data suggests values between 0.1 and 0.3 when it is desirable to detect small changes in whatever process characteristic is being monitored; see, for example, Hunter (1986).

Both Lucas and Saccucci (1990) and Box and Luceno (1997) give the representation

$$Z_i = \lambda \sum_{j=0}^{i-1} (1-\lambda)^j X_{i-j} + (1-\lambda)^i Z_0$$

or an EWMA process. Thus, Zi can be regarded as a moving average of the current and

past values of the control statistics, where the weights on past data fall off exponentially as in a geometric series; and the smaller the value of l , the greater is the influence of the past values. When the Xi are independent and identically distributed with common variance s2, the variance of the control statistics is given by

$$Var(Z_i) = [1-(1-\lambda)^{2i}]\lambda/(2-\lambda)\sigma^2$$

The effect of the starting point soon dissipates and the variance increases quickly to its asymptotic value $\lambda/(2-\lambda)\sigma^2$ as i increases. Control limits are usually based on this asymptotic variance.

The presence of outliers tends to reduce the sensitivity of control chart procedures because the control limits become stretched so that the detection of outliers themselves becomes more difficult; see, for example, Rocke (1989), Tatum (1997) and Langenberg and Iglewicz (1986).

In this chapter, see also Elamir and Seheult (2001), they propose EWMA control charts to monitor the process mean and dispersion using the Gini's mean difference and the sample mean, and also charts based on trimmed versions of the same statistics. The proposed control charts limits are less influenced by extreme observations than classical EWMA control charts, and lead to tighter limits in the presence of out-of-control observations.

Specifically, these control charts and their acronyms are:

- EWMAM: EWMA of the sample mean to monitor the process mean, using Gini's mean difference to estimate the process standard deviation.
- EWMAG:EWMAof the sample Gini's mean difference to monitor process standard deviation.
- EWMATM: EWMA of the sample mean to monitor the process mean, using a trimmed mean of the sample means to estimate the process mean and Gini's mean difference to estimate the process standard deviation.
- EWMATG: EWMA of the sample Gini's mean difference to monitor the process standard deviation using a trimmed mean of the sample Gini's mean differences to estimate the process standard deviation.

REFERENCES

Ali, S. and A. Luceno (1997). Maximum trimmed likelihood estimators: A unified approach.examples and algorithms. Computational Statistics and Data Analysis 25, 251–272.

Amin, R. W. and A. J. Searcy (1991). A nonparametric exponentially weighted moving average control scheme. Communicatin in Statistics-Simulation and Computation 20, 1049–1072.

Amin, R. W. and H. Wollf (1995). The behaviour of EWMA-type quality control schemes in the case of mixture alternatives. Sequential Analysis 14, 157–177.

Barnett, V. and T. Lewis (1994). Outliers in Statistical Data (3rd ed.). New York: John Wiley and Sons.

Box, G. and A. Luceno (1997). Statistical Control by Monitoring and Feedback Adjustment (1st ed.). New York: Wiley.

Chan, L. and L. Rhodin (1980). Robust estimation of location using optimally chosen sample quantile. Technometrics 22, 225–237.

Crowder, S. (1989). Design of exponentially weighted moving average schemes. Journal of Quality Technology 21, 155–162.

David, H. (1981). Order Statistics (2nd ed.). Wiley: New York.

Elamir, E. (2010). Optimal choices of trimming in trimmed L-moments. Applied Mathematical Sciences 4, 2881–2890.

Elamir. E.A.H (2011). Generalizations and Applications of L-moments. Lambert Academic Publishing,ISBN:978-3-8473-0956-7, 2011

Elamir, E. and A. Seheult (2001). Control charts based on linear combinations of order statistics. Journal of Applied Statistics 28, 457–468.

Filliben, J. J. (1969). Simple and Robust Linear Estimation of The Location Parameters of a Symmetric Distribution. Ph.D thesis, Princeton University.

Gan, F. (1995). Joint monitoring of process mean and variance using exponentially weighted moving average control charts. Technometrics 37, 446–453.

Gingras, D. and K. Adamowski (1994). Performance of L-moments and nonparametric flood frequency analysis. Canadian Journal of Civil Engineering 21, 856–862.

Greenwood, J. A., J. M. Landweher, and N. C. Matales (1979). Probability weighted moments: Definition and relation to parameters of several distributions expressable in inverse form. Water Resources Research 15, 1049–1054.

Guttman, N. B., J. Hosking, and J.Wallis (1993). Regional precipitation of quantile values for the continental united states computed from L-moments. Journal Climate 6, 2326–2340.

Hampel, F., E. Ronchetti, P. Rousseeuw, and W. Stahel (1986). Robust Statistics: The Approach Based on Influence Functions (1st ed.). Wiley, New York, NY.

Hawkins, D. (1980). Identification of Outliers (1st ed.). London: Chapman and Hall.

Hosking, J. R. M. and J. Wallis (1987). Parameter and quantile estimation for the generalised pareto distribution. Technometrics 29, 339–349.

Hosking, J. R. M. (1990). L-moments: Analysis and estimation of distributions using linear combinations of statistics. Journal of the Royal Statistical Society B 52, 105–124.

Huber, P. (1981). Robust Statistics (1st ed.). New York: Wiley.

Hunter, J. (1986). The exponentially weighted moving average. Journal of Quality Technology 18, 203–210.

Joiner, B. and J. Rosenblatt (1971). Some properties of the range in samples from Tukey's symmetric lambda distribution. Journal of the American Statistical Association 66, 394–399.

Kirby, W. (1974). Algebraic boundedness of sample statistics. Water Resources Research 10, 220–222.

Langenberg, P. and B. Iglewicz (1986). Trimmed mean $\bar{X}$ and R charts. Journal of Quality Technology 18, 152–161.

Lomoniki (1951). The standard error of Gini's mean difference. Biometrika 34, 151–155.

Lucas, J. and M. Saccucci (1990). Exponentially weighted moving average control schemes: Properties and enhancements. Technometrics 32, 1–13.

Macgregor, J. F. and T. J. Harris (1993). The exponentially weighted moving variance. Journal of Quality Technology 25, 106–118.

Montgomery, D. C. (1996). Introduction to Statistical Quality Control (3rd ed.). New York, Wiley.

Mosteller, F. and J. Tukey (1977). Data Analysis and Regression (1st ed.). Massachusetts: Addison-Wesley, Reading.

Mudholkar, G. and A. Hutson (1998). LQ-moments: Analogs of L-moments. Journal of Statistical Planning and Inference 71, 191–208.

Nair, U. (1936). The standard error of Gini's mean difference. Biometrika 23, 428–436.

Ng, G. and E. Case (1989). Development and evaluation of control charts using exponentially weighted moving averages. Journal of Quality Technology 21, 242–250.

Page, E. (1954). Continuous inspection schemes. Biometrika 41, 100–114.

Pearson, C. (1993). Application of L-moments to maximum river flows. The New Zealand Statistics 28, 02–10.

Pilon, P. and K. Adamowski (1992). The value of regional information to flood frequency analysis using the method of L-moments. Canadian Journal Civil Engineering 19, 137–147.

Ramberg, J. S. and B. W. Schmeiser (1972). An approximate method for generating symmetric random variables. Communication ACM 15, 987–990.

Ramberg, J. S. and B.W. Schmeiser (1974). An approximate method for generating asymmetric random variables. Communication ACM 17, 78–82.

Roberts, S. W. (1959). Control charts based on geometric moving averages. Technometrics 1, 239–250.

Rocke, D. M. (1989). Robust control charts. Technometrics 31, 173–184.

Rousseuw, P. and A. Leroy (1987). Robust Regression and Outlier Detection (1st ed.). Wiley, New York.

Royston, P. (1992). Which measures of skewness and kurtosis are best? Statistics in Medicine 11, 333–343.

Sankarasubramanian, A. and K. Srinivasan (1999). Investigation and comparison of sampling properties of L-moments and conventional moments. Journal of Hydrology 218,13–34.

Sarhan, A. and B. Greenberg (1962). Contribution to Order Statistics (1st ed.). New York: Wiley.

Sillito, G. (1951). Interrelations between certain linear systematic statistics of samples from any continuous distribution. Biometrika 38, 377–382.

Sillito, G. (1969). Derivation of approximations to the inverse distribution function of a continuous univariate population from the order statistics of a sample. Biometrika 56, 641–650.

Tatum, L. G. (1997). Robust estimation of the process standard deviation for control charts. Technometrics 39, 127–141.

Tukey, J. (1960). A survey of sampling from contaminated distributions. In Olkin et al. Ed.), Contributions to probability and statistics, Stanford University Press 1, 448–485.

Tukey, J. (1962). The future of data analysis. Annals of Mathematical Statistics 33, 1–67.

Tukey, J. (1970). Exploratory Data Analysis (1st ed.). Massachusetts: Addison-Wesley, Reading.

Vogel, R. and N. Fennessey (1993). L-moment diagram should replace product moment diagram. Water Resource Research 29, 1745–1752.

Wetherill, G. B. and D. Brown (1991). Statistical Process Control: Theory and Practice (1st ed.). London: Chapman and Hall.